TRAPPED SUBMANIFOLDS IN LORENTZIAN SPACETIMES

SUBVARIEDADES ATRAPADAS EN ESPACIOTIEMPOS LORENTZIANOS

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2018

"Fall in love with the process
and the results will come"

Eric Thomas

Agradecimientos

Estoy feliz de poder dejar por escrito lo agradecida que me siento por haber contado con la ayuda y el apoyo de personas tan maravillosas en la realización de esta tesis.

Estos agradecimientos no pueden empezar de otra forma que con un mayúsculo GRACIAS a mi director de tesis, Luis Alías. Me siento muy afortunada de contar con su infinita ayuda y sus consejos. Es un verdadero placer trabajar con él.

Gracias también a todo el grupo de Geometría y Topología por hacerme sentir una más desde el principio. No me hace falta ser becaria en ningún otro grupo para saber que es una suerte serlo en este.

En esta etapa de doctorado he tenido el placer de realizar diferentes estancias y conocer muchos lugares y gente nueva. Gracias al profesor Ahmad el Soufi, por invitarme a Tours (Francia) y darnos buenos consejos para la mejora de algunos de los resultados de esta memoria. Su inesperado fallecimiento nos dejó no solo sin un gran matemático, sino también sin una gran persona. Ojalá pudiera nombrar a todas las personas que hicieron que recuerde con tanto cariño aquellos meses, gracias a cada una de ellas.

Gracias al profesor Alfonso Romero, por interesarse por mi trabajo y acogerme en su grupo de investigación en Granada en tantas ocasiones. Gracias por su amabilidad, por dedicarme su tiempo y enseñarme tantas matemáticas. Cada vez que voy a Granada me encuentro con gente que me trata tan bien que siempre tengo ganas de volver. Gracias a Santi y especialmente a Jesús por ser tan buenos compañeros en mi último mes allí.

Cuando decidí estudiar matemáticas no me imaginaba que lo haría con compañeros tan geniales: Elvira, Juanjo, Elena, Patricia, Camacho, Pitu, Mariano... Gracias por llenar esos cuatro años de momentos divertidos. Gracias especialmente a Elvira, por seguir ahí aunque nunca le conteste a los whatsapps; eres una gran amiga. Tampoco puedo olvidar aquel curso de doble máster, que tanto nos unió a todos y que me trajo a mi querida Marina desde Castalla, para enseñarme a hablar Valencià y alegrarme el año. Estoy deseando que nos reencontremos.

Estos años de realización de la tesis he tenido la suerte de trabajar en la sala Lázaro Cánovas acompañada de grandes matemáticos y sobre todo amigos: Irene, Antonio, Jose Ginés, Mariano, Mercedes, Jesús... ¡sois todos tan bonicos!. A Irene no puedo dejar de agradecerle todo lo que me ha ayudado y me ayuda. Hacer un doctorado en Matemáticas está bien, pero hacerlo con ella como compañera es otro nivel. Gracias también a Antonio Bueno, que completa la pareja de cuatro que tantos buenos momentos me ha dado en esta última etapa.

Son muchas las personas que, sin formar parte del mundo matemático, me han animado y ayudado en estos años. Gracias al Vammgroup: Aidi, Miri y Mery, porque no hay un solo recuerdo con ellas que no me haga sonreír, y sé que están orgullosas de mi tesis aunque no entiendan ni una palabra. A mi primo Cristian, que fue el primer amigo que me dio la vida y no me imagino otro mejor. Gracias también a Almudena, Emilio, Picón, Juana y los peques, porque a pesar de que el doctorado me hace pasar mucho tiempo sin verlos, cuando quedamos me hacen sentir en casa.

Gracias a Ildefonso por sus incansables palabras de ánimo, por su incalculable ayuda y por ver en mí tantas cosas buenas. Gracias por aparecer en mi vida y querer quedarte.

Por supuesto, si hoy hay una tesis que lleva mi nombre es gracias al apoyo y la ayuda de mi familia. A mis padres, que son los mejores del mundo y eso es un hecho. A mi pequeña Carolina, que es maravillosa y cualquier ratito con ella en el sofá me carga las pilas. Gracias a mi abuela Margarita, por toda su comprensión y por freír cantidades ingentes de almendras para que a mí no me falten.

Gracias a las Matemáticas, por ser tan bellas y dejar que yo lo descubra.

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Notation

The following notations are among the most frequently used throughout the thesis:

p, q	Points
$\mathbf{v}, \mathbf{w}$	Vectors
X, Y, Z, V, W, ζ	Vector fields
f, g, h, u, ϱ	Real functions
M	Lorentzian manifold
Σ	Codimension two spacelike submanifold
ψ	Spacelike immersion
ψ^*	Pull-back via ψ
$\mathfrak{X}^*(\Sigma)$	One-forms on Σ
$\mathfrak{X}(\Sigma)$	Tangent vector fields on Σ
$\mathfrak{X}^\perp(\Sigma)$	Normal vector fields on Σ
$T_p\Sigma$	Tangent vectors to Σ at p
$\mathcal{C}^k(\Sigma)$, $k \in \mathbb{N}$	Functions on Σ with continuous derivative of order k
$\mathcal{C}^\infty(\Sigma)$	Differentiable functions on Σ for all degrees of differentiation
$\mathbb{L}^m$	m-dimensional Lorentz-Minkowski spacetime
$\mathbb{S}^m_1$	m-dimensional unit de Sitter spacetime
$\mathbb{H}^m_1$	m-dimensional unit anti-de Sitter spacetime
$\mathbb{R}^m$	m-dimensional Euclidean space
$\mathbb{S}^m$	m-dimensional unit Euclidean sphere
$\mathbb{H}^m$	m-dimensional unit hyperbolic space

Resumen

La noción de superficie atrapada fue originalmente introducida en Relatividad General por Penrose [46] para el estudio de singularidades y agujeros negros. Estas superficies aparecen también en un contexto puramente matemático en el trabajo de Schoen y Yau [49] sobre la existencia de soluciones a la ecuación de Jang, en relación con su prueba del teorema de la masa positiva. Concretamente, el concepto de superficie atrapada cerrada (donde por cerrada estamos refiriéndonos a una superficie compacta y sin borde) no es útil solamente en cuestiones físicas y desarrollos matemáticos, sino que también tiene diversas aplicaciones: fue un concepto clave, por ejemplo, para el alcance de los teoremas de singularidad, el análisis del colapso gravitacional o el estudio de la hipótesis de censura cósmica [24], [50].

Por otro lado, el interés de matemáticos y físicos en el caso límite de las superficies marginalmente atrapadas se ha visto incrementado en los últimos años. En Relatividad General, las superficies marginalmente atrapadas adquieren gran relevancia a la hora de describir regiones de un espaciotiempo caracterizadas por la existencia de un agujero negro. En particular, los tubos marginalmente atrapados (aquellas hipersuperficies foliadas por superficies marginalmente atrapadas) parecen describir, tal como lo hacen en el modelo de Schwarchild, el horizonte que divide un agujero negro y el resto del espaciotiempo. Por esta razón, el estudio de superficies marginalmente atrapadas es esencial para la determinación de tales horizontes.

Desde un punto de vista estrictamente matemático no hay razón para limitar el estudio del concepto *atrapado* a superficies espaciales, ya que la definición puede extenderse a un contexto más general. Realmente, solo son necesarias dos condiciones: codimensión mayor o igual a dos, y que la métrica inducida en el fibrado normal sea indefinida. En un reciente artículo [34] de Lima, dos Santos y Velásquez estudian la geometría de subvariedades (marginalmente) atrapadas de codimensiones más altas, encontrando condiciones suficientes bajo las que deben ser totalmente umbilicales. Remitimos al lector a la Sección 7 en [17] y

las referencias que allí aparecen para una descripción de algunos de los desarrollos matemáticos recientes en el campo de las subvariedades atrapadas.

Nuestra investigación se desarrolla en el caso de subvariedades espaciales de codimensión dos, las cuales están inmersas en un cierto espaciotiempo M de dimensión $n+2$. Es decir, consideramos una variedad lorentziana M de dimensión $n+2 \geq 4$, temporalmente orientada y una subvariedad espacial Σ de codimensión dos e inmersa en M. En otras palabras, Σ es una variedad conexa, n-dimensional, para la que existe una inmersión $\psi : \Sigma^n \to M^{n+2}$ tal que la métrica inducida sobre Σ es riemanniana. En este contexto, denotamos por $\mathbf{H}$ el campo de vectores curvatura media de la subvariedad. Siguiendo la terminología usada en Relatividad General, decimos que Σ es:

(i) Atrapada futura (pasada) si $\mathbf{H}$ es temporal y apunta hacia el futuro (pasado) sobre Σ.

(ii) Marginalmente atrapada futura (pasada) si $\mathbf{H}$ es nulo y apunta hacia el futuro (pasado) sobre Σ.

(iii) Débilmente atrapada futura (pasada) si $\mathbf{H}$ es causal y apunta hacia el futuro (pasado) sobre Σ.

El caso límite en el que $\mathbf{H} = 0$ corresponde a que Σ sea una subvariedad mínima.

Puesto que Σ es espacial y tiene codimensión dos, cada espacio normal $(T_p\Sigma)^\perp$, $p \in \Sigma$, es temporal y tiene dimensión dos; por tanto, admite dos direcciones nulas, normales a Σ y que apuntan hacia el futuro, las cuales denotamos aquí por ξ y η. Es un hecho conocido que, bajo ciertas hipótesis de orientabilidad (como por ejemplo, si el fibrado normal es trivial) entonces Σ admite una referencia normal nula globalmente definida $\{\xi, \eta\}$ que es única salvo normalización positiva y satisface $\langle \xi, \eta \rangle = -1$. Como es usual en Relatividad, podemos entonces descomponer la segunda forma fundamental utilizando dos formas nulas asociadas a estas dos direcciones nulas ξ y η, de manera que el campo de vectores curvatura media viene dado por

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta,$$

donde θ_ξ y θ_η denotan las llamadas cuvaturas medias nulas (o escalares de expansión nulos) de Σ. Físicamente, θ_ξ (resp. θ_η) mide la divergencia de los rayos de luz que emanan de Σ en la dirección de ξ (resp. η). De hecho, la formulación original de superficie atrapada dada por Penrose [46] fue en términos de los signos o la anulación de las curvaturas medias nulas.

La mayor parte de esta tesis está dedicada a la situación particular en la que la subvariedad espacial de codimensión dos Σ está contenida en una hipersuperficie nula S de un espaciotiempo M. Decimos entonces que Σ factoriza a través de S y escribimos la inmersión como $\psi : \Sigma^n \to S \subset M^{n+2}$. Las hipersuperficies nulas contienen una geometría muy interesante y además juegan un papel relevante en Relatividad General, donde aparecen como horizontes de sucesos de agujeros negros y como horizontes de Cauchy. Sin embargo, a pesar de su importancia, no ha sido hasta los años 1980 cuando se ha empezado a llevar a cabo un estudio continuado de estas subvariedades nulas. Desde entonces, muchos conceptos y resultados del escenario semi-riemanniano se han visto extendidos a este contexto. En [19] y [33] podemos encontrar una visión general sobre el tema.

Recordemos que una hipersuperficie nula en un espaciotiempo M es una subvariedad de codimensión uno, S, embebida en M de manera que la métrica inducida por la métrica lorentziana de M es degenerada. Para el estudio de las hipersuperficies nulas la teoría de las subvariedades no degeneradas falla, ya que existe una intersección no trivial entre sus fibrados tangente y normal. Sin embargo, aunque la métrica inducida es degenerada en S, la familia de subvariedades espaciales contenidas en S la dotan de notables propiedades y, recíprocamente, bajo la hipótesis de que una subvariedad espacial Σ factoriza a través de una hipersuperficie nula S, la geometría intrínseca de Σ se ve limitada. Pensemos por ejemplo en el clásico resultado de Brinkmann en el que enuncia que una variedad riemanniana n-dimensional con $n > 2$ es localmente conformemente llana si, y solo si, existe una inmersión localmente isométrica de esta en el cono de luz del espaciotiempo de Lorentz-Minkowski $(n+2)$-dimensional $\mathbb{L}^{n+2}$ (en [9] podemos encontrar una prueba más actual).

Siguiendo una idea original de Palmas, Palomo y Romero recientemente desarrollada en [44] (véase [45] para el caso previo de superficies espaciales en el cono de luz de $\mathbb{L}^4$), sabemos que si Σ factoriza a través de una hipersuperficie nula, entonces sobre Σ siempre existe de manera natural una referencia normal nula, globalmente definida y que apunta hacia el futuro. Veremos que, cuando consideramos los espaciotiempos de Lorentz-Minkowski o de Sitter, esto nos va a permitir codificar las geometrías extrínseca e intrínseca de la subvariedad en términos de una única función positiva definida sobre Σ. Esta idea está detalladamente desarrollada en los Capítulos 3 y 4.

En esta memoria, antes de adentrarnos en el estudio de las subvariedades espaciales de codimensión dos, empezamos con un capítulo de preliminares, el **Capítulo 2**. Está dedicado a recordar algunos conceptos básicos sobre Geometría lorentziana, subvariedades inmersas en espaciotiempos lorentzianos y, en particular, aquellas que factorizan a través de una hipersuperficie nula. Este segundo capítulo comienza con una sección dedicada a los espaciotiempos lorentzianos, los cuales serán nuestros espacios ambiente durante toda la tesis.

Concretamente, empezamos considerando una variedad lorentziana M y recordando qué es el carácter causal de un vector $\mathbf{v} \in T_pM$. Tras esto, describimos el concepto de orientación temporal y finalmente, en esta sección, incluimos la definición de algunos operadores diferenciales asociados a la métrica: el gradiente, la divergencia, el hessiano y el laplaciano.

Después, nos centramos en las subvariedades espaciales inmersas en variedades lorentzianas, que serán el objeto de nuestro estudio. Aquí establecemos notación básica y también presentamos la segunda forma fundamental, el campo de vectores curvatura media y los operadores forma nulos. En la tercera sección del capítulo tratamos varios tipos notables de subvariedades, los cuales aparecen repetidamente en esta memoria y por eso hemos creído oportuno recordar sus definiciones, así como algunas relaciones y herramientas. De esta forma hablamos de subvariedades atrapadas, variedades estocásticamente completas y parabólicas, de su relación con el principio del máximo débil para el laplaciano, subvariedades totalmente umbilicales, inmersiones conformes e inmersiones isométricas. Concretamente, en esta tesis estudiamos con profundidad las subvariedades atrapadas.

Como hemos adelantado, la mayor parte de nuestro trabajo se desarrolla en el caso en el que una subvariedad espacial de codimensión dos factoriza a través de una hipersuperficie nula. Por esta razón, en la última parte del segundo capítulo nos centramos en estudiar la geometría de tales subvariedades. Bajo esta hipótesis sabemos que existe una referencia normal nula $\{\xi, \eta\}$, globalmente definida, única salvo normalización positiva, que satisface $\langle \xi, \eta \rangle = -1$. Con esto obtenemos una expresión para la segunda forma fundamental, el campo de vectores curvatura media, el tensor curvatura de Riemann, el tensor de Ricci y la curvatura escalar de Σ, todas ellas en términos de ξ y η.

El **Capítulo 3** está dedicado al estudio de la geometría de subvariedades espaciales de codimensión dos que factorizan a través de una hipersuperficie nula del espaciotiempo de Lorentz-Minkowski $(n+2)$-dimensional $\mathbb{L}^{n+2}$. El contenido de este capítulo se corresponde esencialmente con los resultados de nuestro trabajo recogido en [5]. Concretamente nos centramos en la hipersuperficie conocida como el cono de luz del espaciotiempo de Lorentz-Minkowski, Λ, y en la última parte del capítulo obtenemos algunos resultados considerando un hiperplano nulo $\mathcal{L}$. Cabe destacar que es interesante estudiar ambas hipersuperficies nulas ya que tienen diferentes propiedades: por ejemplo, el hiperplano nulo es una hipersuperficie autoparalela mientras que el cono de luz no lo es.

Recordemos que el espaciotiempo de Lorentz-Minkowski $\mathbb{L}^{n+2}$ no es más que el espacio vectorial real $\mathbb{R}^{n+2}$ dotado con la métrica lorentziana

$$\langle , \rangle = -(dx_1)^2 + (dx_2)^2 + \ldots + (dx_{n+2})^2$$

donde estamos tomando $(x_1, x_2, \ldots, x_{n+2})$ como coordenadas. En la primera parte del capítulo consideramos el cono de luz con vértice en el origen $\mathbf{0} \in \mathbb{L}^{n+2}$,

$$\Lambda = \{x \in \mathbb{L}^{n+2} : \langle x, x\rangle = 0, x \neq \mathbf{0}\},$$

y trabajamos en su componente conexa futura

$$\Lambda^+ = \{x \in \mathbb{L}^{n+2} : \langle x, x\rangle = 0, x_1 > 0\}.$$

Denotamos por $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ la inmersión de una subvariedad espacial de codimensión dos, Σ, a través de la componente futura del cono de luz. También definimos la función positiva u como la primera coordenada de la inmersión ψ. En la Sección 3.2, siguiendo las ideas de [44], establecemos las ecuaciones básicas para subvariedades espaciales en el cono de luz, y calculamos su segunda forma fundamental en términos de u y su hessiano (Proposición 3.5 y Proposición 3.6).

Como primera aplicación de nuestro enfoque, la Sección 3.3 está dedicada al estudio de subvariedades totalmente umbilicales que factorizan a través del cono de luz. En particular, en el Teorema 3.9 clasificamos las subvariedades espaciales de codimensión dos en Λ^+, mostrando que, bajo esta hipótesis, existe $\mathbf{a} \in \mathbb{L}^{n+2}$, con $\mathbf{a} \neq 0$ y $\langle \mathbf{a}, \mathbf{a}\rangle = c \in \{-1, 0, 1\}$, y existe $\tau \in \mathbb{R}$, $\tau > 0$, tal que

$$\psi(\Sigma) \subset \Sigma(\mathbf{a}, \tau) = \{p \in \Lambda^+ : \langle p, \mathbf{a}\rangle = \tau\}.$$

En [39], las subvariedades totalmente umbilicales que factorizan a través del cono son caracterizadas por procedimientos completamente diferentes a los aquí mostrados. También en [29] encontramos un estudio sistemático de la umbilicidad y semi-umbilicidad de subvariedades espaciales en el espaciotiempo de Lorentz-Minkowski.

En la Sección 3.4 damos un criterio de compacidad para subvariedades completas en términos del crecimiento de la función positiva u, y obtenemos que toda subvariedad espacial de codimensión dos compacta y que factoriza a través de Λ^+ es entonces conformemente difeomorfa a la esfera euclídea (Proposición 3.12). Es más, probamos que toda subvariedad espacial de codimensión dos en Λ^+ viene dada por un embebimiento explícito que puede escribirse en términos de la función u (Corolario 3.15).

En las dos secciones siguientes nos centramos en el caso en el que la subvariedad

Σ es atrapada. En particular, cuando Σ factoriza a través de la componente futura del cono de luz de $\mathbb{L}^{n+2}$ obtenemos (véase Corolario 3.17) que Σ es atrapada (resp. marginalmente atrapada, débilmente atrapada) y necesariamente pasada si, y solo si, $2u\Delta u - n(1 + \|\nabla u\|^2) > 0$ (resp. $= 0$, ≥ 0) sobre Σ.

Como es sabido [36], no existen subvariedades compactas débilmente atrapadas en $\mathbb{L}^{n+2}$ y, en particular, no existen subvariedades de codimensión dos, compactas y débilmente atrapadas a través del cono de luz de $\mathbb{L}^{n+2}$. En la Proposición 3.22, como aplicación del principio del máximo débil para el laplaciano, extendemos este resultado de no existencia para el caso más general de subvariedades estocásticamente completas bajo la hipótesis de que la función u esté superiormente acotada. En relación a esto, en el Teorema 3.24 también probamos que no existen subvariedades de codimensión dos completas y débilmente atrapadas que factoricen a través del cono de luz de $\mathbb{L}^{n+2}$ para las que la función u satisfaga $u \in L^q(\Sigma)$ para ningún $q > 0$.

Finalizamos este capítulo con el estudio de las subvariedades espaciales de codimensión dos a través del hiperplano nulo dado por

$$\mathcal{L}_{\mathbf{a}} = \{x \in \mathbb{L}^{n+2} : \langle x, \mathbf{a} \rangle = 0, x \neq \mathbf{a}\},$$

donde $\mathbf{a} \in \mathbb{L}^{n+2}$ es un vector nulo. En este caso obtenemos que toda subvariedad $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ es marginalmente atrapada siempre que no se satisfaga $\Delta u = 0$ sobre Σ, donde u está definida como la primera coordenada de la inmersión ψ (Proposición 3.28). Por otra parte, si asumimos la completitud de Σ, en la Proposición 3.29 enunciamos que ésta tiene que ser isométrica al espacio euclídeo $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$ y en el Corolario 3.31 vemos que Σ viene dada por un embebimiento explícito que puede ser escrito en términos de la primera coordenada de la inmersión, u. Como consecuencia de esto podemos dar una caracterización de las subvariedades espaciales de codimensión dos completas que factorizan a través de $\mathcal{L}_{\mathbf{a}}$ y tienen vector curvatura media paralelo (Corolario 3.32).

En el **Capítulo 4** nos interesamos en el estudio de las subvariedades de codimensión dos (marginalmente) atrapadas que están contenidas en ciertas hipersuperficies nulas del espaciotiempo de Sitter $(n+2)$-dimensional,

$$\mathbb{S}_1^{n+2} = \{x \in \mathbb{L}^{n+3} : \langle x, x \rangle = 1\}.$$

En particular, aquí consideramos subvariedades espaciales de codimensión dos que factorizan a través de una de las dos siguientes hipersuperficies nulas, que corresponden a la intersección del espaciotiempo de Sitter con las dos hipersuperficies nulas del espaciotiempo de Lorentz-Minkowski consideradas en el capítulo previo:

(i) la componente futura del cono de luz, que denotaremos por Λ^+, y

(ii) el infinito pasado del espacio conocido como *steady state*, que denotaremos por $\mathcal{J}^-$.

Los resultados que mostramos en este capítulo pueden encontrarse en nuestro artículo [4].

En la Sección 4.2 y la Sección 4.3 consideramos el caso en el que la subvariedad factoriza a través de la componente futura del cono de luz, $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$. En este contexto, obtenemos que la subvariedad Σ es marginalmente atrapada (necesariamente pasada) si, y solo si, la función u satisface la ecuación diferencial

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0 \tag{0.1}$$

sobre Σ, donde u denota la primera coordenada de ψ (Corolario 4.8). Además, también sabemos que toda subvariedad espacial compacta de codimensión dos que factoriza a través de Λ^+ es conformemente difeomorfa a la esfera euclídea $(\mathbb{S}^n, \langle,\rangle_0)$ y viene dada por un embebimiento explícito que puede escribirse en términos de la función u (Corolario 4.15). Asimismo, resolver la ecuación diferencial (0.1) sobre $(\Sigma, \langle,\rangle)$ resulta equivalente a encontrar las soluciones positivas de la ecuación diferencial

$$2f\Delta f - n(1 + \|\nabla f\|^2 - f^2) = 0$$

sobre $\mathbb{S}^n$ con la métrica conforme a la métrica estándar $f^2\langle,\rangle_0$. Como una interesante consecuencia de todo esto, y de manera sorprendente, el problema de caracterizar las subvariedades compactas marginalmente atrapadas en el cono de luz del espaciotiempo de Sitter resulta ser equivalente a resolver el problema de Yamabe en la esfera euclídea, un problema clásico que fue resuelto por Obata [40] en 1971.

Esto nos permite obtener nuestro resultado principal de clasificación y caracterizar todas las subvariedades compactas de codimensión dos que factorizan a través de Λ^+ y son marginalmente atrapadas (Teorema 4.17). En este teorema demostramos que si $\psi : \Sigma \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ es una subvariedad compacta de codimensión dos que factoriza a través de Λ^+ y es marginalmente atrapada (necesariamente pasada); entonces existe un difeomorfismo conforme $\Psi : (\Sigma^n, \langle,\rangle) \to (\mathbb{S}^n, \langle,\rangle_0)$ tal que $\psi = \psi_{\mathbf{b}} \circ \Psi$, donde $f_{\mathbf{b}} : \mathbb{S}^n \to (0, +\infty)$ viene dada por

$$f_{\mathbf{b}}(p) = \frac{1}{\langle p, \mathbf{b}\rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

para cualquier $\mathbf{b} \in \mathbb{R}^{n+1}$ y $\psi_{\mathbf{b}} : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ es el embebimiento

$$\psi_{\mathbf{b}}(p) = (f_{\mathbf{b}}(p), f_{\mathbf{b}}(p)p, 1).$$

En particular, tenemos que Σ es embebida.

Por otra parte, en la Sección 4.4 consideramos el caso de las subvariedades contenidas en el infinito pasado del espacio *steady state*, $\mathcal{J}^-$. En este caso deducimos que toda subvariedad espacial de codimensión dos que factoriza a través de $\mathcal{J}^-$ es marginalmente atrapada siempre que u no satisfaga $\Delta u + nu = 0$ sobre Σ (Proposición 4.24). También observamos que toda subvariedad espacial completa de codimensión dos que factoriza a través de $\mathcal{J}^-$ es necesariamente compacta e isométrica a la esfera euclídea $(\mathbb{S}^n, \langle , \rangle_0)$ y viene dada por un embebimiento explícito que puede ser escrito en términos de la función u (Corolario 4.27). Como consecuencia, en el Corolario 4.28 caracterizamos aquellas subvariedades que, bajo las condiciones anteriores, tienen vector curvatura media paralelo.

La Sección 4.5 está dedicada a establecer un resultado intrínseco de unicidad para las soluciones de la ecuación diferencial (0.1), el cual está motivado por el significado geométrico de tales soluciones. Finalmente, en la Sección 4.6 consideramos el caso más general de las subvariedades espaciales de codimensión dos completas, no compactas y débilmente atrapadas que factorizan a través de Λ^+. Ya conocemos por la Proposición 4.13 que si Σ es una subvariedad de codimensión dos en las condiciones anteriores entonces la función positiva u no puede estar superiormente acotada; es más, tiene que cumplirse

$$\limsup_{r \to +\infty} \frac{u}{r \log(r)} = +\infty$$

con r la distancia riemanniana desde un origen fijado $o \in \Sigma$. En esta dirección, en nuestro Teorema 4.33 probamos que, para tales subvariedades, si el primer valor propio λ_1 del laplaciano Δ es positivo, entonces

$$\left(\int_{\partial B_r} u^q \right)^{-1} \in L^1(+\infty)$$

para cualquier q que satisfaga $0 < q \leq 4\lambda_1/n$. En particular $u \notin L^q(\Sigma)$. La prueba de este resultado es consecuencia de un teorema analítico establecido en el Teorema 4.32, el cual tiene también interés por sí mismo.

En el **Capítulo 5** mostramos una correspondencia natural entre el cono de luz del espaciotiempo de Lorentz-Minkowski y los también llamados conos de luz de los espaciotiempos de Sitter y anti-de Sitter. El estudio desarrollado en este capítulo está incluido en nuestro trabajo [13]. Por simplicidad en la notación, cuando nos estemos refiriendo a cualquiera de los dos espacios, de Sitter o anti-de Sitter indistintamente, lo denotaremos por espaciotiempo (anti)-de Sitter y usaremos la notación $\mathbb{M}_\varepsilon$, donde $\varepsilon = 1$ para el espaciotiempo de Sitter y $\varepsilon = -1$ para el espaciotiempo anti-de Sitter.

En la primera parte del capítulo presentamos el espaciotiempo anti-de Sitter, que viene definido por el subconjunto

$$\mathbb{H}_1^{n+2} = \{x \in \mathbb{R}_2^{n+3} : \langle x, x\rangle = -1\}$$

dotado con la métrica inducida por la de $\mathbb{R}_2^{n+3}$. Aquí, $\mathbb{R}_2^{n+3}$ es el espacio vectorial real $\mathbb{R}^{n+3}$ con coordenadas $(x_0, ..., x_{n+2})$ y métrica de índice $\nu = 2$

$$\langle , \rangle = -(dx_0)^2 - (dx_1)^2 + (dx_2)^2 + \cdots + (dx_{n+2})^2.$$

En esta sección también estudiamos el campo de vectores temporal globalmente definido que nos da la orientación temporal en $\mathbb{H}_1^{n+2}$, y definimos el cono de luz del espaciotiempo (anti)-de Sitter con vértice en $\mathbf{a} \in \mathbb{M}_\varepsilon$ como

$$\widetilde{\Lambda}_{\mathbf{a}} = \{x \in \mathbb{M}_\varepsilon : \langle x - \mathbf{a}, x - \mathbf{a}\rangle = 0, x \neq \mathbf{a}\}.$$

En la Sección 5.2 consideramos las subvariedades espaciales de codimensión dos que factorizan a través del cono de luz del espaciotiempo (anti)-de Sitter, y establecemos una correspondencia entre estas subvariedades y aquellas que factorizan a través del cono de luz del espaciotiempo de Lorentz-Minkowski $\mathbb{L}^{n+2}$. Concretamente en la Proposición 5.4 tenemos que si $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2}$ es una subvariedad espacial de codimensión dos que factoriza a través del cono de luz $\widetilde{\Lambda}_{\mathbf{a}}$, entonces existe una inmersión espacial $\phi : \Sigma^n \to \Lambda \subset \mathbf{a}^\perp$ tal que Σ factoriza a través del cono de luz Λ y hace conmutativo el siguiente diagrama,

$$\begin{tikzcd}
\Lambda \subset \mathbf{a}^\perp \arrow[r, "j"] & \mathbb{E}^{n+3} \\
\Sigma^n \arrow[u, "\phi"] \arrow[r, "\psi"'] & \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2} \arrow[u, "F"']
\end{tikzcd} \tag{0.2}$$

donde $F(x) = x - \mathbf{a}$ induce una isometría totalmente umbilical de $\widetilde{\Lambda}_{\mathbf{a}}$ en Λ y j es la inclusión totalmente geodésica. Aquí, con $\mathbb{E}^{n+3}$ denotamos $\mathbb{L}^{n+3}$ si $\varepsilon = 1$ y $\mathbb{R}_2^{n+3}$ si $\varepsilon = -1$.

En particular, tenemos que las geometrías intrínsecas inducidas por ψ y ϕ sobre Σ son, de hecho, la misma. En este punto nos preguntamos qué podemos decir sobre las geometrías extrínsecas de ψ y ϕ. Con este objetivo en mente obtenemos una relación entre los campos curvatura media correspondientes a ψ y ϕ (Proposición 5.5). Entonces, como consecuencia de esta relación, podemos escribir la curvatura escalar de Σ como (Corolario 5.7)

$$\mathrm{Scal} = n(n-1)(\langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle + \varepsilon). \tag{0.3}$$

Por la correspondencia establecida en el diagrama (0.2), en la última parte de la sección adaptamos algunos resultados del Capítulo 3 y del Capítulo 4 al caso del espaciotiempo (anti)-de Sitter.

La Sección 5.3 está dedicada a las superficies que factorizan a través del cono de luz del espaciotiempo (anti)-de Sitter 4-dimensional. En este caso, obtenemos una fórmula explícita para la curvatura de Gauss en términos de una función altura (Corolario 5.12) y, en la Proposición 5.14, relacionamos el signo de la curvatura de Gauss con la existencia de extremos locales de tal función.

En la última parte del capítulo nos centramos en el caso compacto. En esta ocasión damos un resultado de tipo Liebmann, es decir, probamos que una inmersión espacial compacta con curvatura de Gauss constante en el cono de luz del espaciotiempo (anti)-de Sitter tiene que ser totalmente umbilical (Proposición 5.15). Por otro lado, podemos integrar la fórmula (0.3) para obtener

$$\int_\Sigma \langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle dA = 4\pi - \frac{\varepsilon}{c^2} \operatorname{Area}(\Sigma).$$

Esta fórmula integral es similar al caso de la igualdad en la desigualdad generalizada de Wintgen [51], [53]. Sin embargo, en el caso lorentziano, la desigualdad generalizada de Wintgen no se satisface en general. Finalmente, tratamos con el primer valor propio del operador laplaciano en este tipo de superficies a través del cono de luz del espaciotiempo (anti)-de Sitter. Obtenemos una desigualdad de tipo Reilly (5.8), que usamos para caracterizar las esferas totalmente umbilicales en el cono de luz del espaciotiempo (anti)-de Sitter (Teorema 5.18).

El contenido de esta memoria finaliza con el **Capítulo 6**, donde estudiamos las subvariedades de codimensión dos marginalmente atrapadas en la familia de los espaciotiempos de Robertson-Walker generalizados (GRW). El contenido aquí mostrado se corresponde esencialmente con el de nuestra publicación [3]. Desde un punto de vista geométrico, un espaciotiempo GRW $(n+2)$-dimensional, que aquí denotaremos por $-I \times_\varrho N^{n+1}$, no es más que la variedad producto $I \times N^{n+1}$ de un intervalo real I con una fibra riemanniana $(n+1)$-dimensional $(N^{n+1}, \langle , \rangle_N)$, con la métrica *warped* lorentziana de la forma

$$\langle , \rangle = -dt^2 + \varrho(t)^2 \langle , \rangle_N,$$

donde $\varrho : I \to (0, +\infty)$ es una función regular positiva sobre I, llamada función *warping*. Para todo $\tau \in I$, el *slice* $N_\tau = \{\tau\} \times N$ es una hipersuperficie embebida de $-I \times_\varrho N^{n+1}$ y $\tau \in I \to N_\tau$ determina una foliación de $-I \times_\varrho N^{n+1}$ por hipersuperficies totalmente umbilicales con curvatura media constante $\mathcal{H}(\tau) = -\varrho'(\tau)/\varrho(\tau)$ (podemos encontrar los detalles en la Subsección 6.1.1).

Como consecuencia, las hipersuperficies con curvatura media constante en $(N^{n+1}, \langle , \rangle_N)$ nos dan subvariedades de codimensión dos marginalmente atrapadas en $-I \times_\varrho N^{n+1}$ cuando están contenidas en un *slice* M_τ para un τ apropiado, tal como probaron Flores, Haesen y Ortega en [21, Teorema 2.1] (en [8, Corolario 1] podemos encontrar otro desarrollo diferente por Anciaux y Cipriani). Motivados por este hecho, en este capítulo obtendremos resultados de rigidez que garantizan que, bajo hipótesis apropiadas, las únicas subvariedades de codimensión dos marginamente atrapadas en $-I \times_\varrho N^{n+1}$ son de esta forma (Subsección 6.3.2).

Es en la Sección 6.3 donde presentamos los principales resultados de este capítulo. Estos surgen como aplicación del principio (finito) del máximo para variedades cerradas y, de forma más general, del principio del máximo débil para el laplaciano en el caso de variedades estocásticamente completas. En esta sección empezamos definiendo la función altura h (Definición 6.5) y consideramos la función $u = g(h)$ donde g es una primitiva arbitraria de ϱ. Tras esto, calculamos el operador laplaciano de u,

$$\Delta u = -n(\varrho'(h) - \varrho(h)\langle \mathbf{H}, \partial_t \rangle)$$

y como consecuencia de esta expresión podemos alcanzar algunos resultados interesantes de no existencia para subvariedades débilmente atrapadas en $I \times_\varrho N^{n+1}$ (Subsección 6.3.1).

Seguidamente, en la Subsección 6.3.2, mostramos resultados de rigidez para subvariedades marginalmente atrapadas bajo la hipótesis $(\log(\varrho))'' \leq 0$. Esta hipótesis ha sido muy utilizada por diversos autores para obtener resultados de rigidez para hipersuperficies espaciales con curvatura media constante en espaciotiempos GRW y está fuertemente relacionada con la condición de convergencia temporal (abreviada por sus siglas en inglés TCC) (véase Sección 6.1).

Finalizando el capítulo, la Sección 6.4 está dedicada a mostrar aplicaciones a ciertos casos de espaciotiempos GRW con interés físico. Aquí utilizamos nuestras conclusiones previas para obtener resultados como por ejemplo que, si $(\log(\varrho))'' \leq 0$, entonces las únicas subvariedades cerradas de codimensión dos y marginalmente atrapadas que están embebidas en $-I \times_\varrho \mathbb{R}^{n+1}$ con $\|\mathbf{H}_0\|$

constante son las esferas embebidas dadas por $\{\tau\} \times \mathbb{S}^n(r_\tau)$, con $r_\tau = 1/|\varrho'(\tau)|$ para todo $\tau \in I$ con $\varrho'(\tau) \neq 0$ (Teorema 6.21). Aquí $\mathbf{H}_0$ denota la componente espacial del vector $\mathbf{H}$. Este resultado incluye, por ejemplo, el caso en que el espaciotiempo GRW es el espaciotiempo *steady state* (Corolario 6.22) o el espaciotiempo Einstein-de Sitter (Corolario 6.23).

CHAPTER 1

Introduction

The notion of trapped surface was introduced early in General Relativity by Penrose [46] to study spacetime singularities and black holes. They also arose in a more purely mathematical context in the work of Schoen and Yau [49] about the existence of solutions to the Jang equation, in connection with their proof of the positivity of the mass. Concretely, the concept of closed trapped surface is not only remarkably useful in several physics issues and mathematical progresses, but it also has a lot of diverse applications. For instance, it was a key concept to achieve the singularity theorems, the analysis of gravitational collapse or the study of the cosmic censorship hypotheses [24], [50].

On the other hand, the interest of mathematicians and physicians in the limit case of marginally trapped surfaces has increased in recent years. In General Relativity, marginally trapped surfaces are relevant to describe the regions of a spacetime characterized by the existence of a black hole. In particular, marginally trapped tubes (hypersurfaces foliated by marginally trapped surfaces) may describe, as they do in Schwarzchild model, the horizon that divide the black hole and the rest of the spacetime. For this reason, the study of marginally trapped surfaces is essential for the determination of such horizons.

From a strictly mathematical point of view there is no reason to limit the study to spacelike surfaces, since the definition of trapped can be extended in a more general setting. Actually, only two conditions are needed: codimension greater than or equal to two, and that the induced metric on the normal bundle is indefinite. In the recent paper [34], de Lima, dos Santos and Velásquez studied the geometry of higher codimension (marginally) trapped submanifolds in Lorentzian space forms, and they found sufficient conditions under which they must be totally umbilical. We refer the reader to Section 7 in [17] and references therein for a description of some of the recent mathematical developments in the field of trapped submanifolds.

Our research is developed in the case of codimension two spacelike submanifolds which are immersed in a certain $(n+2)$-dimensional spacetime M, $n \geq 2$. That is, we consider a time-oriented Lorentzian manifold of dimension $n+2 \geq 4$, and let Σ be a codimension two spacelike submanifold immersed in M. In other words, Σ is an n-dimensional connected manifold admitting a smooth immersion $\psi : \Sigma^n \to M^{n+2}$ such that the induced metric on Σ is Riemannian. Denote by $\mathbf{H}$ the mean curvature vector field of the submanifold. Following the standard terminology used in General Relativity, the submanifold Σ is said to be:

(i) *Future (past) trapped* if $\mathbf{H}$ is *timelike and future-pointing (past-pointing)* on Σ.

(ii) *Future (past) marginally trapped* if $\mathbf{H}$ is *null and future-pointing (past-pointing)* on Σ.

(iii) *Future (past) weakly trapped* if $\mathbf{H}$ is *causal and future-pointing (past-pointing)* on Σ.

The extreme case $\mathbf{H} = 0$ corresponds to Σ being a minimal submanifold.

Since Σ is spacelike and it has codimension two, each normal space $(T_p\Sigma)^\perp$, $p \in \Sigma$, is Lorentzian and two dimensional. Hence, it admits two future-pointing null directions normal to Σ, denoted here by ξ and η. It is known that, under suitable orientation assumptions (for instance, if the normal bundle is trivial), Σ admits a globally defined future-pointing normal null frame $\{\xi, \eta\}$, unique up to positive pointwise scaling, satisfying $\langle \xi, \eta \rangle = -1$. As usual in Relativity, we may decompose the second fundamental form into two scalar valued null second forms associated to these null directions ξ and η, so that the mean curvature vector field is given by

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta,$$

where θ_ξ and θ_η denote the null mean curvatures (or null expansion scalars) of Σ. Physically, θ_ξ (resp., θ_η) measures the divergence of the light rays emanating from Σ in the direction of ξ (resp., η). Actually, the original formulation of trapped surfaces given by Penrose [46] was in terms of the signs or the vanishing of the null mean curvatures.

Most of our research is devoted to the particular situation in which the codimension two spacelike submanifold Σ is contained in a null hypersurface S of a spacetime M. We say then that Σ factorizes through S and we write the immersion as $\psi : \Sigma \to S \subset M^{n+2}$. Null hypersurfaces have an interesting geometry and play an important role in General Relativity, where they arise as black hole event horizons and Cauchy horizons. In spite of their relevance, it was not until the decade of 1980 that a regular study of null submanifolds flourished. Since then, many concepts and results from the semi-Riemannian scenario have been extended to this context. In [19] and [33] we can find a broad vision on the subject.

Recall that a null hypersurface in a spacetime M is a smooth codimension one embedded submanifold S of M such that the pullback of the Lorentzian metric of M to S is degenerate. For the study of null hypersurfaces the theory of non-degenerate submanifolds fails. In fact, there is a non trivial intersection between the tangent and the normal bundles of null hypersurfaces. Although the induced metric is degenerate on S, the family of (non-degenerate) spacelike submanifolds through S gives remarkable properties to the null hypersurface and, conversely, under the assumption that a spacelike submanifold Σ factorizes through a fixed null hypersurface S, the intrinsic geometry of Σ becomes limited. For example, recall the classical result by Brinkmann which states that an n-dimensional Riemannian manifold, with $n > 2$, is locally conformally flat if, and only if, it can be locally isometrically immersed in the light cone of the $(n+2)$-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$ (see [9] for a modern proof).

Following an original idea of Palmas, Palomo and Romero recently developed in [44] (see also [45] for the previous case of 2-dimensional spacelike surfaces through the light cone of $\mathbb{L}^4$), we know that if Σ is such a submanifold there always exists a globally and naturally defined future-pointing normal null frame on Σ. We will see that, when working in the Lorentz-Minkowski spacetime or in de Sitter spacetime, this allows us to compute the extrinsic and intrinsic geometry of the submanifold in terms of one single positive function defined on Σ. This idea is highly developed in Chapter 3 and Chapter 4.

In this dissertation, before entering into the study of codimension two spacelike submanifolds, we start with a chapter of preliminaries, **Chapter 2**. It is devoted to recalling some basic concepts about Lorentzian Geometry, submanifolds immersed in Lorentzian spacetimes and, in particular, those which factorize through a null hypersurface. This second chapter begins with a section about Lorentzian spacetimes, which will be our ambient spaces throughout all the thesis. Concretely, we first consider a Lorentzian manifold M, recalling the definition of the causal character of a vector $\mathbf{v} \in T_pM$. Then, we outline the concept of time orientation and finally in this section we include the definition of some differential operators associated to the metric: the gradient, the divergence, the Hessian and the Laplacian.

After that, we deal with spacelike submanifolds in Lorentzian manifolds. They will be the object of our study. Here we stablish some basic notation as well as we present the second fundamental form, the mean curvature vector field and the null shape operators. In the third section of the chapter we deal with various distinguished types of submanifolds. They will appear repeatedly in this thesis, so we think it is worth to remind their definitions and also some relations and tools. In this way we talk about trapped submanifolds, stochastically complete and parabolic manifolds and their relation with the weak maximum principle for the Laplacian. We also deal with totally umbilical submanifolds, conformal immersions and isometric immersions. Concretely, trapped submanifolds will be widely studied in this thesis.

As stated previously, most of our work is developed in the case in which a codimension two spacelike submanifold factorizes through a null hypersurface. For this reason, in the last part of the second chapter we focus on the geometry of such a submanifold. Under this assumption we know that there exists a globally defined future-pointing normal null frame $\{\xi, \eta\}$, unique up to positive pointwise scaling, satisfying $\langle \xi, \eta \rangle = -1$. Assuming that the time orientation of the spacetime is given by a globally defined timelike vector field, we can construct explicitly this normal null frame $\{\xi, \eta\}$. This is how we get an expression for the second fundamental form, the mean curvature vector field, the Riemann curvature tensor, the Ricci tensor and the scalar curvature of Σ, all of them in terms of ξ and η.

Chapter 3 is devoted to the study of the geometry of codimension two spacelike n-submanifolds which factorize through a null hypersurface of the $(n+2)$-dimensional Lorentz-Minkowski spacetime. It corresponds to the research essentially developed in our paper [5]. Concretely, we focus on the null hypersurface known as the light cone of the Lorentz-Minkowski spacetime, Λ. At the end of the chapter we obtain some results considering a null hyperplane $\mathcal{L}$ as the null hypersurface. Observe that it is worth taking into account both null hypersurfaces since they have different properties: for instance, the null hyperplane is an autoparallel hypersurface, while the light cone is not.

Recall that the Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$ is nothing but the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle , \rangle = -(dx_1)^2 + (dx_2)^2 + \ldots + (dx_{n+2})^2$$

where we are taking $(x_1, x_2, \ldots, x_{n+2})$ as coordinates. In the first part of the chapter, we consider the light cone with vertex at the origin $\mathbf{0} \in \mathbb{L}^{n+2}$,

$$\Lambda = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x \neq \mathbf{0}\},$$

and we work in its future component

$$\Lambda^+ = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x_1 > 0\}.$$

We denote by $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ the immersion of a codimension two spacelike submanifold Σ through the future component of the light cone. We also define the positive function u as the first coordinate of the immersion ψ. In Section 3.2, following the procedure in [44], we establish the basic equations for spacelike submanifolds through the light cone and compute their second fundamental form in terms of u and its Hessian (Proposition 3.5 and Proposition 3.6).

As a first application of our approach, Section 3.3 is devoted to the study of totally umbilical submanifolds factorizing through the light cone. In particular, in Theorem 3.9 we classify codimension two totally umbilical spacelike submanifolds in Λ^+. We show that, under these assumptions, there exist $\mathbf{a} \in \mathbb{L}^{n+2}$, $\mathbf{a} \neq 0$, $\langle \mathbf{a}, \mathbf{a} \rangle = c \in \{-1, 0, 1\}$, and $\tau \in \mathbb{R}$, $\tau > 0$, such that

$$\psi(\Sigma) \subset \Sigma(\mathbf{a}, \tau) = \{p \in \Lambda^+ : \langle p, \mathbf{a} \rangle = \tau\}.$$

In [39], totally umbilical submanifolds through the light cone are characterized by a completely different procedure. See also [29] for a systematic study of the umbilicity and semi-umbilicity of spacelike submanifolds of the Lorentz-Minkowski space.

In Section 3.4 we give a compactness criterion for complete submanifolds in terms of the growth of the positive function u. We obtain that every codimension two compact spacelike submanifold contained in Λ^+ is conformally diffeomorphic to the round sphere (Proposition 3.12). Even more, we prove that every codimension two compact spacelike submanifold through Λ^+ is given by an explicit embedding which can be written in terms of the single function u (Corollary 3.15).

In the two following sections we focus on the case where the submanifold Σ is trapped. In particular, when Σ factorizes through the future component of the light cone of $\mathbb{L}^{n+2}$ we obtain that (see Corollary 3.17) Σ is (necessarily past) trapped (resp. marginally trapped, weakly trapped) if, and only if, $2u\Delta u - n(1 + \|\nabla u\|^2) > 0$ (resp. $= 0$, ≥ 0) on Σ.

It is already known [36] that there exists no compact weakly trapped submanifold in $\mathbb{L}^{n+2}$ and, in particular, there is no codimension two compact weakly trapped submanifold through the light cone of $\mathbb{L}^{n+2}$. In Proposition 3.22, and as an application of the weak maximum principle for the Laplacian, we extend this non-existence result to the more general case of stochastically complete submanifolds, under the assumption that the function u is bounded from above.

Related to this, in Theorem 3.24 we also prove that there exists no codimension two complete weakly trapped submanifold Σ through the light cone of $\mathbb{L}^{n+2}$ for which the function u satisfies $u \in L^q(\Sigma)$, for any $q > 0$.

To finish the chapter we study codimension two spacelike submanifolds through the null hyperplane given by

$$\mathcal{L}_{\mathbf{a}} = \{x \in \mathbb{L}^{n+2} : \langle x, \mathbf{a} \rangle = 0, x \neq \mathbf{a}\},$$

where $\mathbf{a} \in \mathbb{L}^{n+2}$ is a null vector. In this case we obtain that every submanifold $\psi \colon \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ is always marginally trapped, except at points where $\Delta u = 0$, being u defined as the first coordinate of the immersion ψ (see Proposition 3.28). On the other hand, if we assume the completeness of Σ, in Proposition 3.29 we state that it has to be isometric to the Euclidean space $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$. Moreover, in Corollary 3.31 we see that it is given by an explicit embedding which can be written in terms of u. As a consequence of this we achieve a characterization for complete codimension two spacelike submanifolds which factorize through $\mathcal{L}_{\mathbf{a}}$ and have parallel mean curvature vector (see Corollary 3.32).

In **Chapter 4** we are interested in the study of codimension two (marginally) trapped submanifolds which factorize through certain null hypersurfaces of the $(n+2)$-dimensional de Sitter spacetime $\mathbb{S}_1^{n+2}$,

$$\mathbb{S}_1^{n+2} = \{x \in \mathbb{L}^{n+3} : \langle x, x \rangle = 1\}.$$

In particular, here we consider codimension two spacelike submanifolds which factorize through one of the two following null hypersurfaces, which correspond to the intersection of de Sitter spacetime with the two null hypersurfaces of the Lorentz-Minkowski spacetime considered in the previous chapter:

(i) the future component of the light cone, which we will denote by Λ^+, and

(ii) the past infinite of the steady state space, which we will denote by $\mathcal{J}^-$.

The results included in this chapter can be found in our paper [4].

In Section 4.2 and Section 4.3 we consider the case of submanifolds which factorize through the future component of the light cone $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$. In that case, we derive that a submanifold Σ is (necessarily past) marginally trapped if, and only if, the function u satisfies the differential equation

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0 \tag{1.1}$$

on Σ, where u denote the first coordinate of ψ (see Corollary 4.8). Moreover, we also obtain that every codimension two compact spacelike submanifold which factorizes through Λ^+ is conformally diffeomorphic to the round sphere $(\mathbb{S}^n, \langle , \rangle_0)$ and is given by an explicit embedding which can be written in terms of the single function u (Corollary 4.15). Besides, solving the differential equation (1.1) on $(\Sigma, \langle , \rangle)$ becomes equivalent to finding the positive solutions of the differential equation

$$2f\Delta f - n(1 + \|\nabla f\|^2 - f^2) = 0$$

on $\mathbb{S}^n$ endowed with the conformal metric $f^2\langle , \rangle_0$. As a nice consequence of all this, and quite surprisingly, the problem of characterizing compact marginally trapped submanifolds through the light cone of de Sitter spacetime becomes equivalent to solving the Yamabe problem on the unit round sphere, a classical problem that was solved by Obata [40] in 1971. This allows us to obtain our main classification result and to characterize all codimension two compact marginally trapped submanifolds factorizing through Λ^+ (see Theorem 4.17). In this theorem we consider $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ a codimension two compact (necessarily past) marginally trapped spacelike immersed submanifold factorizing through Λ^+. Then there exists a conformal diffeomorphism $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ such that $\psi = \psi_{\mathbf{b}} \circ \Psi$, where $f_{\mathbf{b}} : \mathbb{S}^n \to (0, +\infty)$ is

$$f_{\mathbf{b}}(p) = \frac{1}{\langle p, \mathbf{b}\rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

for any $\mathbf{b} \in \mathbb{R}^{n+1}$ and $\psi_{\mathbf{b}} : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ is the embedding

$$\psi_{\mathbf{b}}(p) = (f_{\mathbf{b}}(p), f_{\mathbf{b}}(p)p, 1).$$

In particular, Σ is embedded.

On the other hand, in Section 4.4 we consider the case of submanifolds contained in the past infinite of the steady state space $\mathcal{J}^-$. In this case we deduce that every codimension two spacelike submanifold Σ factorizing through $\mathcal{J}^-$ is always marginally trapped except at points where $\Delta u + nu = 0$ on Σ (Proposition 4.24). Moreover, we also observe that every codimension two complete spacelike submanifold which factorizes through $\mathcal{J}^-$ is necessarily compact and isometric to the round sphere $(\mathbb{S}^n, \langle , \rangle_0)$. Even more, we know that it is given by an explicit embedding which can be written in terms of the single function u (Corollary 4.27). As a consequence, in Corollary 4.28 we characterize those having parallel mean curvature vector.

Section 4.5 is devoted to establish an intrinsic uniqueness result for the solutions of the differential equation (1.1) which is motivated by the geometric meaning of its solutions. Finally, in Section 4.6 we consider the more general case of codimension two complete, non-compact, weakly trapped spacelike submanifolds factorizing through Λ^+. We already know from Proposition 4.13 that if Σ is a codimension two complete, non-compact, weakly trapped submanifold through Λ^+, then the positive function u cannot be bounded above. Going further, it has to be

$$\limsup_{r\to+\infty} \frac{u}{r\log(r)} = +\infty$$

with r the Riemannian distance from a fixed origin $o \in \Sigma$. In this direction, in our Theorem 4.33 we prove that, for such submanifolds, if the first eigenvalue λ_1 of the Laplacian Δ is positive then

$$\left(\int_{\partial B_r} u^q\right)^{-1} \in L^1(+\infty)$$

for any q satisfying $0 < q \leq 4\lambda_1/n$. In particular $u \notin L^q(\Sigma)$. The proof of this result will be a consequence of an intrinsic analytical result established in Theorem 4.32, with its own interest.

In **Chapter 5** we show a natural correspondence between the light cone of the Lorentz-Minkowski spacetime and the also called light cones of de Sitter and anti-de Sitter spacetimes. The study developed in this chapter is included in our work [13]. For the sake of simplicity, when we are referring to both de Sitter and anti-de Sitter spacetime we name them (anti)-de Sitter spacetime, and we use the notation $\mathbb{M}_\varepsilon$ where $\varepsilon = 1$ for de Sitter spacetime and $\varepsilon = -1$ for anti-de Sitter spacetime.

In the first part of the chapter we introduce (anti)-de Sitter spacetime, which is defined as the subset

$$\mathbb{H}_1^{n+2} = \{x \in \mathbb{R}_2^{n+3} : \langle x, x\rangle = -1\}$$

endowed with the induced metric from $\mathbb{R}_2^{n+3}$. Here, $\mathbb{R}_2^{n+3}$ is the real vector space $\mathbb{R}^{n+3}$ with canonical coordinates $(x_0, ..., x_{n+2})$ and metric

$$\langle , \rangle = -(dx_0)^2 - (dx_1)^2 + (dx_2)^2 + \cdots + (dx_{n+2})^2.$$

In this section we also study the globally defined timelike vector field which gives us the time orientation on $\mathbb{H}_1^{n+2}$ and we present the light cone of (anti)-de Sitter spacetime with vertex at $\mathbf{a} \in \mathbb{M}_\varepsilon$ as

$$\widetilde{\Lambda}_{\mathbf{a}} = \{x \in \mathbb{M}_\varepsilon : \langle x - \mathbf{a}, x - \mathbf{a}\rangle = 0, x \neq \mathbf{a}\}.$$

In Section 5.2 we consider codimension two spacelike submanifolds which factorize through a light cone of (anti)-de Sitter spacetime and we stablish a correspondence between these submanifolds and those which factorize through the light cone in Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$. Concretely, in Proposition 5.4 we have that, if $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^{n+2}_{\varepsilon}$ is a codimension two spacelike submanifold which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$, then there exists a spacelike immersion $\phi : \Sigma^n \to \Lambda \subset \mathbf{a}^{\perp}$ such that Σ factorizes through the light cone Λ and that makes the following diagram commutative,

$$\begin{tikzcd} \Lambda \subset \mathbf{a}^{\perp} \arrow[r, "j"] & \mathbb{E}^{n+3} \\ \Sigma^n \arrow[u, "\phi"] \arrow[r, "\psi"'] & \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^{n+2}_{\varepsilon} \arrow[u, "F"'] \end{tikzcd} \tag{1.2}$$

where $F(x) = x - \mathbf{a}$ induces a (totally umbilical) isometry from $\widetilde{\Lambda}_{\mathbf{a}}$ to Λ and j is the totally geodesic inclusion. In particular, we have that the intrinsic geometries on Σ induced by ψ and ϕ are, in fact, the same. Here, $\mathbb{E}^{n+3}$ stands for $\mathbb{L}^{n+3}$ if $\varepsilon = 1$ and for $\mathbb{R}^{n+3}_2$ if $\varepsilon = -1$.

At this point we wonder what we can say about the extrinsic geometries of ψ and ϕ. With this aim in mind we have obtained a relation between the mean curvature vector fields corresponding to ψ and ϕ (Proposition 5.6). Then, as a consequence of this relation, we can write the scalar curvature of Σ as follows (Corollary 5.7)

$$\mathrm{Scal} = n(n-1)(\langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle + \varepsilon). \tag{1.3}$$

By means of the correspondence stablished in the diagram (1.2), in the last part of the section we can adapt some results of Chapter 3 and Chapter 4 to the case of (anti)-de Sitter spacetime.

Section 5.3 is devoted to surfaces which factorize through a light cone of the 4-dimensional (anti)-de Sitter spacetime. In this case we obtain an explicit formula for the Gauss curvature in terms of a height function (Corollary 5.12) and, in Proposition 5.14, we relate the sign of the Gauss curvature with the existence of local extreme points of such height function.

In the last part of the chapter we focus on the compact case. In this instance we give a Liebmann-type result for these surfaces, i.e., we have that a compact spacelike immersion with constant Gauss curvature through a light cone of (anti)-de Sitter spacetime has to be totally umbilical (Proposition 5.15). On the other hand, formula (1.3) can be integrated to give

$$\int_\Sigma \langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle dA = 4\pi - \frac{\varepsilon}{c^2}\,\mathsf{Area}(\Sigma).$$

This integral formula looks very similar to the equality case of the generalized Wintgen inequality [51], [53]. However, in the Lorentzian setting, the generalized Wintgen inequality is not satisfied in general. Finally, we deal with the first eigenvalue of the Laplacian operator of such kind of surfaces through a light cone in (anti)-de Sitter spacetime. We obtain a Reilly type inequality (5.8), which we use to characterize the total umbilical round spheres in a light cone of (anti)-de Sitter spacetime (Theorem 5.18).

This dissertation ends with **Chapter 6**. In this chapter we are focused on the study of codimension two marginally trapped submanifolds in the family of general Robertson-Walker (GRW) spacetimes. The content shown here corresponds essentially to that of our publication [3]. From the geometric viewpoint, an $(n+2)$-dimensional GRW spacetime, denoted here by $-I \times_{\varrho} N^{n+1}$, is nothing but the product manifold $I \times N^{n+1}$ of an open real interval I with an $(n+1)$-dimensional Riemannian fiber $(N^{n+1}, \langle , \rangle_N)$, endowed with a Lorentzian warped metric of the form

$$\langle , \rangle = -dt^2 + \varrho(t)^2 \langle , \rangle_N,$$

where $\varrho : I \to (0, +\infty)$ is a positive smooth function on I, called the warping function. For every $\tau \in I$, the slice $N_\tau = \{\tau\} \times N$ is an embedded spacelike hypersurface of $-I \times_{\varrho} N^{n+1}$ and $\tau \in I \to N_\tau$ determines a foliation of $-I \times_{\varrho} N^{n+1}$ by totally umbilical spacelike hypersurfaces with constant mean curvature $\mathcal{H}(\tau) = -\varrho'(\tau)/\varrho(\tau)$ (see Subsection 6.1.1 for the details).

As a consequence, hypersurfaces with constant mean curvature in $(N^{n+1}, \langle , \rangle_N)$ produce codimension two marginally trapped submanifolds of $-I \times_{\varrho} N^{n+1}$ when contained in a slice M_τ at an appropriate τ, as proved by Flores, Haesen and Ortega in [21, Theorem 2.1] (see also [8, Corollary 1] for an alternative approach by Anciaux and Cipriani). Motivated by this fact, in this chapter we obtain some rigidity results which guarantee that, under appropriate hypothesis, the only codimension two marginally trapped submanifolds in $-I \times_{\varrho} N^{n+1}$ are of this form (see Subsection 6.3.2).

It is in Section 6.3 where we present the main results of the chapter. They will be an application of the (finite) maximum principle for closed manifolds and, more generally, of the weak maximum principle for the Laplacian for stochastically complete manifolds. In this section we start defining the height function h (see Definition 6.5) and we consider the function $u = g(h)$, where g is an arbitrary primitive of ϱ. Then, we compute the Laplacian of u,

$$\Delta u = -n(\varrho'(h) - \varrho(h)\langle \mathbf{H}, \partial_t \rangle)$$

and, as a consequence of this expression, we can derive some interesting non-existence results for weakly trapped submanifolds in $-I \times_{\varrho} N^{n+1}$ (see Subsection 6.3.1).

After that, in Subsection 6.3.2 we derive some rigidity results for marginally trapped submanifolds, assuming that $(\log(\varrho))'' \leq 0$. This hypothesis has been widely used by several authors to obtain rigidity results for spacelike hypersurfaces with constant mean curvature in GRW spacetimes and it is closely related to the timelike convergence condition (TCC) (see Section 6.1).

Finishing the chapter, Section 6.4 is devoted to give applications to some cases of GRW spacetimes with physical relevance. We use our previous conclusions to obtain some results. For instance, we prove that, if $(\log(\varrho))'' \leq 0$, the only closed marginally trapped n-submanifolds which are embedded in $-I \times_{\varrho} \mathbb{R}^{n+1}$ with $\|\mathbf{H}_0\|$ constant are the embedded n-spheres given as $\{\tau\} \times \mathbb{S}^n(r_\tau)$, with $r_\tau = 1/|\varrho'(\tau)|$ for every $\tau \in I$ with $\varrho'(\tau) \neq 0$ (Theorem 6.21). Here $\mathbf{H}_0$ stands for the spacelike component of the vector $\mathbf{H}$. This result includes, for instance, the case where the spacetime is the steady state spacetime (Corollary 6.22) or the Einstein-de Sitter spacetime (Corollary 6.23).

CHAPTER 2

Preliminaries

2.1 Lorentzian geometry

Our aim in this section is to introduce the Lorentzian spacetimes, which will be our ambient spaces throughout this thesis. To do this, let us remind some basic, and surely well known for the reader, definitions and properties, such as the concept of Lorentzian manifold.

Definition 2.1. An m-dimensional Lorentzian manifold is a pair $(M, \langle , \rangle)$ where M is an m-dimensional manifold and $\langle , \rangle$ is a metric with index $\nu = 1$.

Recall that the index ν of a symmetric bilinear form $\langle , \rangle$ on a vector space V is the largest integer that is the dimension of a subspace $W \subset V$ on which $\langle , \rangle_{|W}$ is negative definite.

For the sake of simplicity, if $(M, \langle , \rangle)$ is an m-dimensional Lorentzian manifold, we denote it just by M, taking in mind that there is a Lorentzian metric defined on M and indicating the dimension only when relevant. Within this context, and due to the index of the metric, we can distinguish the following types of vectors on M.

Definition 2.2. Let $\mathbf{v} \in T_pM$ be a tangent vector at a point $p \in M$. We say that $\mathbf{v}$ is

(i) spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = 0$,

(ii) timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$,

(iii) null (or lightlike) if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and $\mathbf{v} \neq 0$, and

(iv) causal if $\langle \mathbf{v}, \mathbf{v} \rangle \leq 0$ and $\mathbf{v} \neq 0$.

This is called the causal character of the vector $\mathbf{v}$.

These definitions can be extended to the case of a tangent vector field $X \in \mathfrak{X}(M)$ considering that X is spacelike (resp. timelike, null, causal) if $X_p := X(p)$ is a spacelike (resp. timelike, null, causal) vector at every point $p \in M$.

It is well known that the subset of timelike vectors (resp. causal, null if $m \geq 2$) has two connected parts and each one of these parts will be called timelike cone (resp. causal cone, null cone). From this fact we can give the following definition.

Definition 2.3. A time orientation on a Lorentzian manifold is a smooth choice of one of the timelike cones. The chosen cone will be called future cone and the other one, past cone.

Hence, we say that timelike vectors which are in the future cone are future-pointing, and those which are in the past cone are past-pointing. Next lemma ([43, Lemma 5.29]) gives us a criterion to distinguish whether or not two timelike vectors are in the same timelike cone.

Lemma 2.4. Two timelike vectors $\mathbf{v}, \mathbf{w} \in T_pM$, for $p \in M$, lie in the same timelike cone if, and only if, $\langle \mathbf{v}, \mathbf{w} \rangle < 0$.

At this point, we are in position to present Lorentzian spacetimes.

Definition 2.5. A Lorentzian spacetime is a Lorentzian manifold M where we have chosen a time orientation at every point $p \in M$.

Observe that one condition that ensures the time orientability of a Lorentzian manifold is the existence of a globally defined timelike vector field $\mathbf{e} \in \mathfrak{X}(M)$. Indeed, setting $p \in M$, we define the future cone at p as the component of the timelike cone containing $\mathbf{e}_p$. In this case, we say that the time orientation on M is given by $\mathbf{e}$ and we are now able to orient every causal vector on M with the following criterion.

Definition 2.6. Let M be a Lorentzian spacetime whose time orientation is given by the globally defined timelike vector field $\mathbf{e}$. We say that a causal vector $\mathbf{v} \in T_pM$, for $p \in M$, is pointing to the future if, and only if, $\langle \mathbf{v}, \mathbf{e}_p \rangle < 0$. We say that $\mathbf{v}$ is pointing to the past otherwise.

Below we include the definition of some differential operators associated to the metric as well as some of their properties which will be widely used all along this thesis. To do this, we denote by ∇ the Levi-Civita connection on the Lorentzian manifold M.

Definition 2.7. The gradient, ∇f, of a function $f \in C^\infty(M)$ is the vector field metrically equivalent to the differential $df \in \mathfrak{X}^*(M)$. It is defined by the relation

$$\langle \nabla f, X \rangle = X(f) = df(X)$$

for every $X \in \mathfrak{X}(M)$.

Observe that in terms of a coordinate system $df = \sum_i (\partial f/\partial x^i) dx^i$ and hence,

$$\nabla f = \sum_{i,j} g^{i,j} \frac{\partial f}{\partial x^i} \partial_j, \tag{2.1}$$

where $[g^{i,j}]$ stands for the inverse matrix of the metric of M.

Likewise, if $f : M \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two smooth functions, we can compute the gradient of the composition $g \circ f : M \to \mathbb{R}$,

$$\nabla(g \circ f) = g'(f) \nabla f.$$

Notice that we use the same notation for both the Levi-Civita connection and the gradient operator. We are sure that this will not cause any misunderstanding since the gradient acts on functions, while the Levi-Civita connection does on vector fields.

Definition 2.8. The divergence, $\operatorname{div}(X)$, of a vector field $X \in \mathfrak{X}(M)$ is the smooth function defined by

$$\operatorname{div}(X) = \operatorname{tr}(Y \mapsto \nabla_Y X).$$

where tr denotes the trace with respect to the metric of M.

Now, let $f : M \to \mathbb{R}$ be a smooth function and $X \in \mathfrak{X}(M)$. Then, it follows

$$\operatorname{div}(fX) = f \operatorname{div}(X) + \langle \nabla f, X \rangle.$$

Definition 2.9. The Hessian, $\nabla^2 f$, of a function $f \in C^\infty(M)$ is its second covariant differential

$$\nabla^2 f = \nabla(\nabla f).$$

On the other hand, for any $f \in \mathcal{C}^\infty(M)$ we denote by Hess_f the symmetric $(0,2)$

tensor on M which is metrically equivalent to the Hessian operator,

$$\begin{aligned}\mathrm{Hess}_f : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathbb{R} \\ (X, Y) &\to \langle \nabla_X(\nabla f), Y \rangle.\end{aligned} \tag{2.2}$$

Finally in this section, we introduce the Laplacian operator.

Definition 2.10. The Laplacian, Δf, of a function $f \in C^\infty(M)$ is the smooth function defined as the divergence of its gradient, that is,

$$\Delta f = \mathrm{div}(\nabla f).$$

The Laplacian has coordinate expression

$$\begin{aligned}\Delta f &= \sum_{i,j} g^{i,j}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{i,j} \frac{\partial f}{\partial x^k}\right) \\ &= \frac{1}{\sqrt{|\det(g_{i,j})|}} \sum_i \frac{\partial}{\partial x^i}\left(\sqrt{|\det(g_{i,j})|} \sum_j g^{i,j} \frac{\partial f}{\partial x^j}\right),\end{aligned} \tag{2.3}$$

where $\Gamma^k_{i,j}$ denote the Christoffel symbols associated to the metric of M.

Taking $g : \mathbb{R} \to \mathbb{R}$ and $f : M \to \mathbb{R}$ two smooth functions, the Laplacian of the composition $g \circ f : M \to \mathbb{R}$ is given by

$$\Delta(g \circ f) = g'(f)\Delta f + g''(f)\|\nabla f\|^2.$$

2.2 Spacelike submanifolds in Lorentzian manifolds

Once we know what our ambient spaces are, we may present the submanifolds, whose geometry will be studied throughout this thesis. More concretely, we will be focused on codimension two spacelike submanifolds of Lorentzian spacetimes.

Definition 2.11. An n-dimensional submanifold Σ of an $m(\geq n)$-dimensional Lorentzian manifold M is an n-dimensional manifold such that there exists an immersion $\psi : \Sigma \to M$. The integer $m - n$ is said to be the codimension of the submanifold.

The immersion $\psi : \Sigma \to M$ provides a metric on Σ. It is called the induced metric and it is defined by

$$\langle X, Y\rangle = \psi^*(\langle X, Y\rangle_M) = \langle d\psi(X), d\psi(Y)\rangle_M$$

where $X, Y \in \mathfrak{X}(\Sigma)$, and $\langle , \rangle_M$ stands for the metric of M. Depending on this induced metric, we can find the following notable kinds of submanifolds.

Definition 2.12. Let Σ be a submanifold immersed in a Lorentzian manifold M. We say that:

(i) Σ is spacelike if the induced metric has index $\nu = 0$, that is, if it is a Riemannian metric on Σ.

(ii) Σ is timelike if the induced metric has index $\nu = 1$, that is, if it is a Lorentzian metric on Σ.

(iii) Σ is null (or lightlike) if the induced metric is degenerate on Σ.

Our research is developed in the case of spacelike submanifolds. From an intrinsic point of view they are nothing but Riemannian manifolds, and this allows us to use the strong tools of the Riemannian geometry.

Let $\psi : \Sigma \to M$ be a spacelike submanifold, and let us denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of M and Σ respectively. We also denote by $\nabla^\perp$ the normal connection of Σ in M. Then, the Gauss and Weingarten formulas of ψ are given by

$$\overline{\nabla}_X Y = \nabla_X Y - \mathrm{II}(X, Y) \tag{2.4}$$

and

$$\overline{\nabla}_X \zeta = A_\zeta X + \nabla^\perp_X \zeta, \tag{2.5}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$ and normal vector field $\zeta \in \mathfrak{X}^\perp(\Sigma)$. Observe that here, and throughout all the thesis,

$$\mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^\perp(\Sigma)$$

stands for the vector valued second fundamental form of the submanifold, and that we are following for II the usual convention in Relativity (and opposite to the one usually taken in Differential Geometry). With this convention, for instance, the mean curvature vector field of a round sphere in Euclidean space points outwards. Moreover, for every normal vector $\zeta \in \mathfrak{X}^\perp(\Sigma)$, A_ζ denotes the shape operator (or Weingarten endomorphism) associated to ζ; that is, the symmetric operator $A_\zeta : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ given by

$$\langle A_\zeta X, Y\rangle = \langle \mathrm{II}(X, Y), \zeta\rangle.$$

As usual, we also define the mean curvature vector field of the submanifold by

$$\mathbf{H} = \frac{1}{n}\mathrm{tr}(\mathrm{II}) \in \mathfrak{X}^{\perp}(\Sigma),$$

where tr stands for the trace with respect to the induced metric on Σ.

2.3 Some distinguished types of submanifolds

As stated previously, we are going to focus our attention on codimension two spacelike submanifolds. In this way, our principal results are devoted to the study of some of them with special characteristics. Namely, we will talk repeatedly about trapped, stochastically complete and totally umbilical submanifolds, as well as about parabolicity, isometries or conformal immersions. In this section we introduce all these concepts, together with some relations and tools which will be used further on.

2.3.1 Trapped submanifolds

Following the standard terminology used in General Relativity, and depending on the causal character of the mean curvature vector field, we have the following definition.

Definition 2.13. Let $\psi : \Sigma^n \to M^{n+2}$ be a codimension two spacelike submanifold. We say that Σ is

(i) future (past) trapped if $\mathbf{H}$ is timelike and future-pointing (past-pointing) on Σ.

(ii) Future (past) marginally trapped if $\mathbf{H}$ is null and future-pointing (past-pointing) on Σ.

(iii) Future (past) weakly trapped if $\mathbf{H}$ is causal and future-pointing (past-pointing) on Σ.

The extreme condition $\mathbf{H} = 0$ corresponds to a minimal submanifold.

A particular case occurs when, working with a codimension two submanifold Σ, we are able to find a global normal null frame $\{\xi, \eta\}$, that is, ξ and η are two globally defined normal null vector fields on Σ. Within this context, take ξ and η both futute pointing, with $\langle \xi, \eta \rangle = -1$, and let A_ξ and A_η be the associated shape operators. Then, we can define the null mean curvatures as follows.

Definition 2.14. The null mean curvatures (or null expansion scalars) associated to ξ and η are, respectively, the functions

$$\theta_\xi = \frac{1}{n}\mathrm{tr}(A_\xi) \quad \text{and} \quad \theta_\eta = \frac{1}{n}\mathrm{tr}(A_\eta),$$

where tr stands for the trace with respect to the induced metric on Σ

Since $A_\zeta X = (\overline{\nabla}_X \zeta)^\top$ for every normal vector field $\zeta \in \mathfrak{X}^\perp(\Sigma)$, it follows that

$$\theta_\xi = \frac{1}{n}\mathrm{div}_\Sigma \xi \quad \text{and} \quad \theta_\eta = \frac{1}{n}\mathrm{div}_\Sigma \eta.$$

That means that, physically, θ_ξ (resp., θ_η) measures the divergence of the light rays emanating from Σ in the direction of ξ (resp., η). In terms of these null mean curvatures the mean curvature vector field is written as

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta, \tag{2.6}$$

and its norm gets the expression

$$\langle \mathbf{H}, \mathbf{H} \rangle = -2\theta_\xi \theta_\eta. \tag{2.7}$$

This yields that we can rewrite the concepts of Definition 2.13 using this globally defined normal null frame $\{\xi, \eta\}$.

(i) Σ is a trapped submanifold if, and only if, either both $\theta_\xi < 0$ and $\theta_\eta < 0$ (future trapped), or both $\theta_\xi > 0$ and $\theta_\eta > 0$ (past trapped).

(ii) Σ is a marginally trapped submanifold if, and only if, either $\theta_\xi = 0$ and $\theta_\eta \neq 0$ (future marginally trapped if $\theta_\eta < 0$ and past marginally trapped if $\theta_\eta > 0$), or $\theta_\xi \neq 0$ and $\theta_\eta = 0$ (future marginally trapped if $\theta_\xi < 0$ and past marginally trapped if $\theta_\xi > 0$).

(iii) Σ is a weakly trapped submanifold if, and only if, either both $\theta_\xi \leq 0$ and $\theta_\eta \leq 0$ with $\theta_\xi^2 + \theta_\eta^2 > 0$ (future weakly trapped), or both $\theta_\xi \geq 0$ and $\theta_\eta \geq 0$ with $\theta_\xi^2 + \theta_\eta^2 > 0$ (past weakly trapped).

This was the original formulation of trapped surfaces given by Penrose [46] in terms of the signs or the vanishing of the null mean curvatures. Trapped surfaces were introduced to study spacetime singularities and black holes. They also arose in a more purely ma-thematical context in the work of Schoen and Yau [49] about the existence of solutions to the Jang equation, in connection with their proof of the positivity of the mass.

2.3.2 Stochastically completeness and the weak maximum principle for the Laplacian

In this part of the section we deal with stochastically complete manifolds. They will be highly regarded for our results since the weak maximum principle for the Laplacian (which we will also formulate here) holds in such kind of manifolds.

Definition 2.15. A (non necessarily complete) Riemannian manifold Σ is said to be stochastically complete if its Brownian motion is stochastically complete, that is, if the probability of a particle to be found in the state space is constantly equal to 1.

In other words, if Σ is stochastically complete, then

$$\int_{\Sigma} p(x,y,t)dy = 1 \quad \text{for any } (x,t) \in \Sigma \times (0,+\infty),$$

where $p(x,y,t)$ is the heat kernel of the Laplacian operator. Actually, for any open $\Omega \subset \Sigma$, $\int_{\Omega} p(x,y,t)dy$ is the probability that a random path starting at x lies in Ω at finite time t. Hence, $\int_{\Sigma} p(x,y,t)dy < 1$ means that there is a positive probability that a random path will reach infinity in finite time t.

Later in this work, we will make use of a weaker version of the Omori-Yau maximum principle. Following the terminology introduced by Pigola, Rigoli and Setti in [48], the Omori-Yau maximum principle is said to hold on a Riemannian manifold Σ if, for any smooth function $u \in \mathcal{C}^2(\Sigma)$ with $u^* = \sup_{\Sigma} u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties

$$(i) \quad u(p_k) > u^* - \frac{1}{k}, \quad (ii) \quad \|\nabla u(p_k)\| < \frac{1}{k}, \quad \text{and} \quad (iii) \quad \Delta u(p_k) < \frac{1}{k}$$

for every $k \in \mathbb{N}$. Equivalently, for any smooth function $u \in \mathcal{C}^2(\Sigma)$ with $u_* = \inf_{\Sigma} u > -\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ such that

$$(i) \quad u(p_k) < u_* + \frac{1}{k}, \quad (ii) \quad \|\nabla u(p_k)\| < \frac{1}{k}, \quad \text{and} \quad (iii) \quad \Delta u(p_k) > -\frac{1}{k}$$

for every $k \in \mathbb{N}$. In this sense, the classical result given by Omori and Yau in [42, 55] states that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below. More generally, as shown by Pigola, Rigoli and Setti [48, Example 1.13], a sufficiently controlled decay of the radial Ricci curvature of the form

$$\mathrm{Ric}_\Sigma(\nabla r, \nabla r) \geq -C^2 G(r) \tag{2.8}$$

suffices to imply the validity of the Omori-Yau maximum principle, where r is the distance function on Σ to a fixed point, C is a positive constant, and $G\colon [0,+\infty) \to \mathbb{R}$ is a smooth function satisfying

$$\text{(i)}\ \ G(0) > 0,\ \ \text{(ii)}\ \ G'(t) \geq 0,\ \ \text{(iii)}\ \ \int_0^{+\infty} 1/\sqrt{G(t)} = +\infty \ \text{ and}$$

$$\text{(iv)}\ \ \limsup_{t\to+\infty} tG(\sqrt{t})/G(t) < +\infty.$$

In particular, and following the terminology introduced by Bessa and Costa in [11], the Omori-Yau maximum principle holds on a complete Riemannian manifold whose Ricci curvature has strong quadratic decay [16], that is, with

$$\mathrm{Ric}_\Sigma \geq -C^2(1 + r^2 \log^2(2+r)).$$

Following again the terminology introduced in [48], the weak maximum principle for the Laplacian is said to hold on a (non necessarily complete) Riemannian manifold Σ if, for any smooth function $u \in \mathcal{C}^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties

$$(i) \quad u(p_k) > u^* - \frac{1}{k}, \quad \text{and} \quad (ii) \quad \Delta u(p_k) < \frac{1}{k} \tag{2.9}$$

for every $k \in \mathbb{N}$. Equivalently, for any smooth function $u \in \mathcal{C}^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ such that

$$(i) \quad u(p_k) < u_* + \frac{1}{k}, \quad \text{and} \quad (ii) \quad \Delta u(p_k) > -\frac{1}{k} \tag{2.10}$$

for every $k \in \mathbb{N}$. As proved by Pigola, Rigoli and Setti [47], the fact that the weak maximum principle for the Laplacian holds on Σ is equivalent to the stochastic completeness of the manifold (see also [48, Theorem 3.1]). This proves to be very interesting because, for instance, every closed (meaning compact and without boundary) Riemannian manifold is stochastically complete. Furthermore, in next subsection we will see that parabolicity is strongly related with the stochastic completeness.

2.3.3 Parabolicity and the weak maximum principle for the Laplacian

Let us start by recalling the definition of a parabolic manifold.

Definition 2.16. A (non necessarily complete) Riemannian manifold Σ is parabolic if every subharmonic function on Σ which is bounded from above must be constant.

That is, Σ is a parabolic manifold if

$$\left.\begin{aligned} u \in C^2(\Sigma) \\ \Delta u \geq 0 \\ \sup_\Sigma u < +\infty \end{aligned}\right\} \implies u \text{ constant.}$$

There are several interesting geometric conditions which imply the parabolicity of a Riemannian manifold Σ. For instance, in dimension $n = 2$ parabolicity is strongly related to the behaviour of the Gaussian curvature. In this context, from a classical result by Ahlfors [1] and Blanc-Fiala-Huber [28] it is well known that every complete Riemannian surface Σ with non-negative Gaussian curvature is parabolic. More generally, every complete Riemannian surface Σ with finite total curvature is parabolic [28] (see also [35, Section 10]). In this direction, if the Gaussian curvature K of a complete Riemannian surface Σ satisfies $K \geq -1/(r^2 \log r)$ for r sufficiently large, where r is the distance function to a fixed point, then Σ is parabolic [22].

In higher dimension, parabolicity is quite different and it seems not to have any clear relation with sectional curvature. Actually, the Euclidean space $\mathbb{R}^n$ is parabolic if, and only if, $n \leq 2$. On the other hand, the product of a parabolic Riemannian manifold with a closed Riemannian manifold is always parabolic. In particular, the Riemannian product $\mathbb{R}^2 \times \Sigma$ of $\mathbb{R}^2$ with any closed Riemannian manifold Σ is always parabolic, regardless of its dimension [31]. In this case, the volume of the geodesic balls in $\mathbb{R}^2 \times \Sigma$ of radius r still grows like r^2 and the manifold $\mathbb{R}^2 \times \Sigma$ is parabolic. For the hyperbolic plane, the volume of the geodesic balls of radius r grows exponentially and it is not parabolic. This leads one to suspect that the rate of volume growth of geodesic balls may be significant to the parabolicity, as observed by Karp [30], who proved that every complete Riemannian manifold with moderate volume growth is parabolic.

If we now turn to the stochastic completeness of Σ, we see that it is equivalent (among other conditions) to the fact that for every $\lambda > 0$, the only non-negative bounded smooth solution u of $\Delta u \geq \lambda u$ on Σ is the constant $u = 0$. In particular, every parabolic Riemannian manifold is stochastically complete. Recall that, as a consequence, the weak maximum principle for the Laplacian holds on every parabolic Riemannian manifold (see also [23, Corollary 6.4]), and so, it is also true for closed Riemannian manifolds since every closed Riemannian manifold is parabolic.

2.3.4 More notable kinds of immersions

In this section we briefly remind some definitions. We may point out that, unlike in the case of stochastically complete and parabolic submanifolds, all the properties here introduced (as well as the concept of trapped submanifold) are extrinsic. That is, they depend not only on the submanifold Σ, but also on the immersion ψ. We start with totally umbilical submanifolds.

Definition 2.17. An n-dimensional submanifold $\psi : \Sigma \to M$ is said to be totally umbilical if it is umbilical with respect to all possible normal directions $\zeta \in \mathfrak{X}^{\perp}(\Sigma)$. That is, for every $\zeta \in \mathfrak{X}^{\perp}(\Sigma)$ there exists a smooth function $\lambda_\zeta \in \mathcal{C}^{\infty}(\Sigma)$ such that
$$A_\zeta = \lambda_\zeta I$$
where A_ζ is the shape operator associated to ζ.

Now, observing the induced metric on Σ we can find the following immersions.

Definition 2.18. Let $\psi : \Sigma \to M$ be an immersion between two manifolds $(\Sigma, \langle,\rangle_\Sigma)$ and $(M, \langle,\rangle_M)$. We say that Σ is a conformal submanifold (and ψ a conformal immersion) if the induced metric satisfies
$$\psi^*(\langle,\rangle_M) = \lambda^2 \langle,\rangle_\Sigma$$
for some function $\lambda \in \mathcal{C}^{\infty}(\Sigma)$, $\lambda > 0$. The positive function λ is called the conformal factor.

If $\psi : \Sigma \to M$ is a conformal immersion with conformal factor λ, we can obtain the following relations for the operators defined in the preceding section with respect to $\langle,\rangle_\Sigma$ and $\widetilde{\langle,\rangle} = \lambda^2\langle,\rangle_\Sigma$.
$$\lambda^2 \widetilde{\nabla} f = \nabla f, \tag{2.11}$$
$$\widetilde{\nabla}^2 f = \nabla^2 f - \frac{1}{\lambda}(d\lambda \otimes df + df \otimes d\lambda) + \frac{1}{\lambda}\langle \nabla\lambda, \nabla f\rangle \langle,\rangle, \tag{2.12}$$

$$\widetilde{\mathrm{div}}(X) = \mathrm{div}(X) + n\frac{d\lambda}{\lambda} \tag{2.13}$$

and

$$\lambda\widetilde{\Delta}f = \Delta f + \frac{n-2}{\lambda}\langle\nabla\lambda, \nabla f\rangle, \tag{2.14}$$

where f is a function $f \in \mathcal{C}^\infty(\Sigma)$ and $X \in \mathfrak{X}(\Sigma)$. In the previous definition, if λ is a constant c we say that Σ is a homothetic to M with coefficient c^2. In the case that $c = 1$ we have the following.

Definition 2.19. Let $\psi : \Sigma \to M$ be an immersion between two manifolds $(\Sigma, \langle,\rangle_\Sigma)$ and $(M, \langle,\rangle_M)$ with dim$(\Sigma) =$ dim(M). We say that ψ is an isometry if the induced metric coincides with the original metric on Σ, that is,

$$\psi^*(\langle,\rangle_M) = \langle,\rangle_\Sigma.$$

In this case we say that Σ and M are isometric manifolds.

2.4 Codimension two spacelike submanifolds through a null hypersurface

In the following chapters we study the special case when the codimension two spacelike submanifold Σ is contained in a null hypersurface S of a specific Lorentzian spacetime M, that is, when the immersion $\psi : \Sigma^n \to M^{n+2}$ satisfies $\psi(\Sigma) \subset S \subset M$ being M a known Lorentzian spacetime. In this situation we say that the submanifold Σ factorizes through the null hypersurface S, and we write the immersion as $\psi : \Sigma^n \to S \subset M^{n+2}$.

In this section we wonder what we can say about the geometry of the submanifold Σ when the ambient spacetime M is an arbitrary Lorentzian one. Let us assume that the orientation on M is given by the globally defined timelike vector field T and let $\psi : \Sigma^n \to M^{n+2}$ be a codimension two spacelike submanifold on M which factorizes through a null hypersurface of the spacetime. In this case, there always exists a globally defined null vector field $\xi \in \mathfrak{X}^\perp(\Sigma)$ which is normal to the submanifold and future-pointing. In this section we will see that, when this happens, we are able to build a globally defined future-pointing normal null frame $\{\xi, \eta\}$.

With this aim in mind, let ξ be a globally defined normal null vector field which is future-pointing. That is, $\xi \in \mathfrak{X}^{\perp}(\Sigma)$ satisfies

$$\langle \xi, \xi \rangle = 0 \quad \text{and} \quad \langle \xi, T \rangle < 0.$$

We decompose T as

$$T = T^{\top} + T^{\perp},$$

where $T^{\top}$ is the component which is tangent to Σ and $T^{\perp}$ is the component which is normal to Σ. Observe that

$$-1 = \langle T, T \rangle = \|T^{\top}\|^2 + \langle T^{\perp}, T^{\perp} \rangle,$$

and thus,

$$\langle T^{\perp}, T^{\perp} \rangle = -1 - \|T^{\top}\|^2 \leq -1 < 0.$$

Now, we consider the globally defined unit timelike vector field

$$\nu = \frac{T^{\perp}}{\sqrt{1 + \|T^{\top}\|^2}}, \tag{2.15}$$

which is also normal to Σ and future-pointing. Therefore $\langle \xi, \nu \rangle < 0$ and the vector field

$$\eta = -\frac{1}{2\langle \xi, \nu \rangle^2}\xi - \frac{1}{\langle \xi, \nu \rangle}\nu \tag{2.16}$$

provides us a second globally defined normal null vector field along the submanifold which is future-pointing and satisfies $\langle \xi, \eta \rangle = -1$.

From now on in this section, we study the geometry of Σ in terms of $\{\xi, \eta\}$. For every $X, Y \in \mathfrak{X}(\Sigma)$, the second fundamental form can be written as

$$\mathrm{II}(X, Y) = \alpha(X, Y)\xi + \beta(X, Y)\eta$$

with α, $\beta \in \mathcal{T}_2^0(\Sigma)$, that is, $\alpha, \beta : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathcal{C}^{\infty}(\Sigma)$. Computing

$$\langle \mathrm{II}(X, Y), \xi \rangle = \beta(X, Y)\langle \eta, \xi \rangle = -\beta(X, Y),$$

and we have

$$\beta(X, Y) = -\langle A_{\xi}X, Y \rangle$$

where A_{ξ} is the shape operator associated to ξ.

On the other hand,

$$\langle \mathrm{II}(X, Y), \eta \rangle = \alpha(X, Y)\langle \xi, \eta \rangle = -\alpha(X, Y),$$

and then

$$\alpha(X,Y) = -\langle A_\eta X, Y\rangle,$$

being A_η the shape operator associated to η. Henceforth, the second fundamental form is amened to read as follows

$$\mathrm{II}(X,Y) = -\langle A_\eta X, Y\rangle \xi - \langle A_\xi X, Y\rangle \eta. \tag{2.17}$$

Taking traces from this expression we obtain that the mean curvature field is given by

$$\mathbf{H} = -\frac{1}{n}(\mathrm{tr}(A_\eta)\xi + \mathrm{tr}(A_\xi)\eta)$$

and its norm is

$$\langle \mathbf{H}, \mathbf{H}\rangle = -\frac{2}{n^2}\mathrm{tr}(A_\xi)\mathrm{tr}(A_\eta).$$

Observe that this identity is equivalent to the one showed in (2.7) in terms of the null mean curvatures θ_ξ and θ_η.

In order to know the expression of the Riemann curvature tensor of Σ, R, and the Ricci curvature of Σ, Ric, we will use the Gauss equation

$$\begin{aligned}\langle R(V,W)X,Y\rangle &= \langle \bar{R}(V,W)X,Y\rangle + \langle \mathrm{II}(V,X), \mathrm{II}(W,Y)\rangle \\ &\quad - \langle \mathrm{II}(V,Y), \mathrm{II}(W,X)\rangle \\ &= \langle \bar{R}(V,W)X,Y\rangle + \langle A_{\mathrm{II}(V,X)}W, Y\rangle \\ &\quad - \langle A_{\mathrm{II}(V,Y)}W, X\rangle,\end{aligned} \tag{2.18}$$

where $X,Y,V,W \in \mathfrak{X}(\Sigma)$ and $\bar{R}$ stands for the Riemann curvature tensor of M. Recall that in our convention

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

Thus, we have

$$R(X,Y)V = (\bar{R}(X,Y)V)^\top + A_{\mathrm{II}(X,V)}Y - A_{\mathrm{II}(Y,V)}X,$$

and tracing this,

$$\begin{aligned}\mathrm{Ric}(X,Y) = \mathrm{tr}(R) &= \sum_{i=1}^n \langle R(X,e_i)Y, e_i\rangle \\ &= \sum_{i=1}^n \left(\langle (\bar{R}(X,e_i)Y)^\top, e_i\rangle + \langle A_{\mathrm{II}(X,Y)}e_i, e_i\rangle - \langle A_{\mathrm{II}(e_i,Y)}X, e_i\rangle\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left(\langle \bar{R}(X, e_i)Y, e_i \rangle + \langle \mathrm{II}(X, Y), \mathrm{II}(e_i, e_i) \rangle - \langle \mathrm{II}(X, e_i), \mathrm{II}(Y, e_i) \rangle \right) \\
&= \sum_{i=1}^{n} \left(\langle \bar{R}(X, e_i)Y, e_i \rangle + \langle \mathrm{II}(X, Y), n\mathbf{H} \rangle - \langle \mathrm{II}(X, e_i), \mathrm{II}(Y, e_i) \rangle \right).
\end{aligned}
$$

Here, $\{e_1, \dots, e_n\}$ is a local orthonormal frame on Σ. Taking into account what we have obtained in (2.17),

$$
\langle \mathrm{II}(X, e_i), \mathrm{II}(Y, e_i) \rangle = -2 \langle A_\eta X, e_i \rangle \langle A_\xi Y, e_i \rangle
$$

and from here,

$$
\sum_{i=1}^{n} \langle \mathrm{II}(X, e_i), \mathrm{II}(Y, e_i) \rangle = -2 \langle (A_\xi \circ A_\eta) X, Y \rangle.
$$

So far we have reached

$$
\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \langle \bar{R}(X, e_i)Y, e_i \rangle + \langle \mathrm{II}(X, Y), n\mathbf{H} \rangle + 2 \langle (A_\xi \circ A_\eta) X, Y \rangle.
$$

We study now the first term in previous identity, $\sum_{i=1}^{n} \langle \bar{R}(X, e_i)Y, e_i \rangle$. To do this we denote by $\overline{\mathrm{Ric}}$ the Ricci curvature of M and recall that, from [43, Lemma 3.52],

$$
\overline{\mathrm{Ric}}(X, Y) = \langle \bar{R}(X, \nu_1)Y, \nu_1 \rangle - \langle \bar{R}(X, \nu_2)Y, \nu_2 \rangle + \sum_{i=1}^{n} \langle \bar{R}(X, e_i)Y, e_i \rangle, \quad (2.19)
$$

where $\{\nu_1, \nu_2\}$ is an orthonormal frame of $\mathfrak{X}^\perp(\Sigma)$ such that $\langle \nu_1, \nu_1 \rangle = 1$ and $\langle \nu_2, \nu_2 \rangle = -1$. We can construct these two vector fields as

$$
\nu_1 = \frac{1}{\sqrt{2}}(\xi - \eta) \qquad \text{and} \qquad \nu_2 = \frac{1}{\sqrt{2}}(\xi + \eta),
$$

and with this choice

$$
\langle \bar{R}(X, \nu_1)Y, \nu_1 \rangle - \langle \bar{R}(X, \nu_2)Y, \nu_2 \rangle = -\langle \bar{R}(X, \xi)Y, \eta \rangle - \langle \bar{R}(X, \eta)Y, \xi \rangle.
$$

From here, identity (2.19) becomes

$$\sum_{i=1}^{n} \langle \bar{R}(X, e_i)Y, e_i \rangle = \overline{\mathrm{Ric}}(X, Y) + \langle \bar{R}(X, \xi)Y, \eta \rangle + \langle \bar{R}(X, \eta)Y, \xi \rangle,$$

and hence, we can write the Ricci tensor of our submanifold Σ as follows

$$\begin{aligned} \mathrm{Ric}(X, Y) = \overline{\mathrm{Ric}}(X, Y) &+ \langle \bar{R}(X, \xi)Y, \eta \rangle + \langle \bar{R}(X, \eta)Y, \xi \rangle \\ &+ n\langle \mathrm{II}(X, Y), \mathbf{H} \rangle + 2\langle (A_\xi \circ A_\eta)X, Y \rangle. \end{aligned} \tag{2.20}$$

Finally in this section we compute the scalar curvature of Σ, defined as usual by

$$\mathrm{Scal} = \mathrm{tr}(\mathrm{Ric}).$$

For this, let us start by taking $\{e_1, \ldots, e_n\}$ a local orthonormal frame on Σ. Then, using equation (2.20) and some basic properties of the Riemann curvature tensor R,

$$\begin{aligned} \mathrm{Scal} = &\sum_{i=1}^{n} \overline{\mathrm{Ric}}(e_i, e_i) + 2\sum_{i=1}^{n} \langle \bar{R}(e_i, \xi)e_i, \eta \rangle \\ &+ n\mathrm{tr}(A_{\mathbf{H}}) + 2\mathrm{tr}(A_\xi \circ A_\eta). \end{aligned} \tag{2.21}$$

On the other hand, taking into account the definition of ν_1 and ν_2 we can write the scalar curvature of M as

$$\begin{aligned} \overline{\mathrm{Scal}} = &\sum_{i=1}^{n} \overline{\mathrm{Ric}}(e_i, e_i) + \overline{\mathrm{Ric}}(\nu_1, \nu_1) - \overline{\mathrm{Ric}}(\nu_2, \nu_2) \\ = &\sum_{i=1}^{n} \overline{\mathrm{Ric}}(e_i, e_i) - 2\overline{\mathrm{Ric}}(\xi, \eta). \end{aligned}$$

Following some straightforward computations we have that

$$\overline{\mathrm{Ric}}(\xi, \eta) = \sum_{i=1}^{n} \langle \bar{R}(e_i, \xi)e_i, \eta \rangle - \langle \bar{R}(\xi, \eta)\eta, \xi \rangle,$$

and then,

$$\overline{\mathrm{Scal}} = \sum_{i=1}^{n} \overline{\mathrm{Ric}}(e_i, e_i) - 2\sum_{i=1}^{n} \langle \bar{R}(e_i, \xi)e_i, \eta \rangle + 2\langle \bar{R}(\xi, \eta)\eta, \xi \rangle.$$

Putting this into (2.21) it follows

$$\mathrm{Scal} = \overline{\mathrm{Scal}} + 4\overline{\mathrm{Ric}}(\xi, \eta) + 2\langle \bar{R}(\xi, \eta)\eta, \xi\rangle + n\mathrm{tr}(A_{\mathbf{H}}) + 2\mathrm{tr}(A_\xi \circ A_\eta). \quad (2.22)$$

On the other hand, since $\{\xi_p, \eta_p\}$ is a basis of $(T_p\Sigma)^\perp$ at any $p \in \Sigma$, the sectional curvature spanned by ξ and η is written as

$$\bar{K}(\xi \wedge \eta) = \frac{\langle \bar{R}(\xi, \eta)\xi, \eta\rangle}{\bar{Q}(\xi, \eta)},$$

where $\bar{Q}(\xi, \eta) = \langle \xi, \xi\rangle\langle \eta, \eta\rangle - \langle \xi, \eta\rangle^2 = -1$. Then, $\bar{K}(\xi \wedge \eta) = -\langle \bar{R}(\xi, \eta)\xi, \eta\rangle = \langle \bar{R}(\xi, \eta)\eta, \xi\rangle$. Inserting this in (2.22) and using that $\mathrm{tr}(A_{\mathbf{H}}) = n\langle \mathbf{H}, \mathbf{H}\rangle$, the expression we finally get is

$$\mathrm{Scal} = \overline{\mathrm{Scal}} + 4\overline{\mathrm{Ric}}(\xi, \eta) + 2\bar{K}(\xi \wedge \eta) + n^2\langle \mathbf{H}, \mathbf{H}\rangle + 2\mathrm{tr}(A_\xi \circ A_\eta).$$

Next proposition summarizes our computations.

Proposition 2.20. Let $\psi : \Sigma^n \to M^{n+2}$ be a codimension two spacelike submanifold such that there exists a normal null frame $\{\xi, \eta\}$ with $\langle \xi, \eta\rangle = -1$. Then, with the previous notation, we have for Σ:

(i) the second fundamental form

$$\mathrm{II}(X, Y) = -\langle A_\eta X, Y\rangle\xi - \langle A_\xi X, Y\rangle\eta, \quad (2.23)$$

(ii) the mean curvature vector field

$$\mathbf{H} = \frac{1}{n}\mathrm{tr}(\mathrm{II}) = -\frac{1}{n}\mathrm{tr}(A_\eta)\xi - \frac{1}{n}\mathrm{tr}(A_\xi)\eta, \quad (2.24)$$

(iii) the Riemann curvature tensor

$$R(X, Y)V = (\bar{R}(X, Y)V)^\top + A_{\mathrm{II}(X,V)}Y - A_{\mathrm{II}(Y,V)}X, \quad (2.25)$$

(iv) the Ricci tensor

$$\begin{aligned}\mathrm{Ric}(X, Y) = \overline{\mathrm{Ric}}(X, Y) &+ \langle \bar{R}(X, \xi)Y, \eta\rangle + \langle \bar{R}(X, \eta)Y, \xi\rangle \\ &+ n\langle \mathrm{II}(X, Y), \mathbf{H}\rangle + 2\langle (A_\xi \circ A_\eta)X, Y\rangle,\end{aligned} \quad (2.26)$$

(v) and the scalar curvature

$$\mathrm{Scal} = \overline{\mathrm{Scal}} + 4\overline{\mathrm{Ric}}(\xi, \eta) + 2\bar{K}(\xi \wedge \eta) + n^2\langle \mathbf{H}, \mathbf{H}\rangle + 2\mathrm{tr}(A_\xi \circ A_\eta). \quad (2.27)$$

CHAPTER 3

Codimension two spacelike submanifolds through a null hypersurface of the Lorentz-Minkowski spacetime

This chapter, which corresponds to the research essentially developed in our paper [5], is devoted to the case in which $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ is a codimension two spacelike submanifold which factorizes through a null hypersurface of the Lorentz-Minkowski spacetime. Notice that the computations and expressions obtained in Section 2.4 for an arbitrary spacetime will be very useful tools in this study.

3.1 Preliminaries

For the purpose of fixing the notation, and also recalling some basic concepts, we start defining our ambient space, the Lorentz-Minkowski spacetime of dimension $n+2$.

Definition 3.1. The $(n+2)$-dimensional Lorentz-Minkowski space, $\mathbb{L}^{n+2}$, is defined as the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle , \rangle = -(dx_1)^2 + (dx_2)^2 + \ldots + (dx_{n+2})^2,$$

where $(x_1, x_2, \ldots, x_{n+2})$ are the canonical coordinates of $\mathbb{R}^{n+2}$.

We will denote by $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{L}^{n+2}$ and we will consider on this space the time orientation induced by the globally defined timelike vector field

$$\mathbf{e}_1 = (1, 0, \dots, 0).$$

Let Σ be a codimension two spacelike submanifold in the Lorentz-Minkowski spacetime with Levi-Civita connection ∇ and which factorizes through a null hypersurface of $\mathbb{L}^{n+2}$. In this case, and as we will see later, there always exists a globally defined null vector field $\xi \in \mathfrak{X}^{\perp}(\Sigma)$ which is normal to the submanifold and future-pointing. Let $\mathbf{e}_1^{\perp}$ denote the normal component of $\mathbf{e}_1$ along the submanifold, that is, for every $p \in \Sigma$, we have the following orthogonal decomposition

$$\mathbf{e}_1 = \mathbf{e}_1^{\top} + \mathbf{e}_1^{\perp}, \tag{3.1}$$

where $\mathbf{e}_1^{\top} \in \mathfrak{X}(\Sigma)$ is tangent to Σ and $\mathbf{e}_1^{\perp} \in \mathfrak{X}^{\perp}(\Sigma)$ is normal to Σ. In particular

$$\langle \mathbf{e}_1^{\perp}, \mathbf{e}_1^{\perp} \rangle = -1 - \|\mathbf{e}_1^{\top}\|^2 \leq -1 < 0$$

and the vector field ν defined in (2.15) is given by

$$\nu = \frac{\mathbf{e}_1^{\perp}}{\|\mathbf{e}_1^{\perp}\|} = \frac{\mathbf{e}_1^{\perp}}{\sqrt{1 + \|\mathbf{e}_1^{\top}\|^2}}. \tag{3.2}$$

In this way, ν determines along the submanifold a globally defined unit timelike vector field which is normal to Σ and future-pointing. In particular, $\langle \xi, \nu \rangle < 0$ and, using formula (2.16), the vector field

$$\eta = \frac{\langle \mathbf{e}_1^{\perp}, \mathbf{e}_1^{\perp} \rangle}{2\langle \xi, \mathbf{e}_1 \rangle^2} \xi - \frac{1}{\langle \xi, \mathbf{e}_1 \rangle} \mathbf{e}_1^{\perp} \tag{3.3}$$

provides us another globally defined normal null vector field along the submanifold which is future-pointing and satisfies $\langle \xi, \eta \rangle = -1$.

It follows from the Gauss equation (2.18) that the Riemann curvature tensor R of Σ is given by

$$R(X, Y)Z = A_{\mathrm{II}(X,Z)} Y - A_{\mathrm{II}(Y,Z)} X$$

for any tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$. In particular, in this context, formulas (2.26) and (2.27) imply that the Ricci and the scalar curvature of Σ are given by

$$\mathrm{Ric}(X, Y) = n\langle \mathbf{H}, \mathrm{II}(X, Y) \rangle + 2\langle (A_\xi \circ A_\eta) X, Y \rangle \tag{3.4}$$

and

$$\mathrm{Scal} = n^2 \langle \mathbf{H}, \mathbf{H} \rangle + 2\mathrm{tr}(A_\xi \circ A_\eta). \tag{3.5}$$

The two other fundamental equations of the submanifold Σ are the Codazzi and the Ricci equations. In our case, the Codazzi equation of Σ is given by

$$(\nabla_X \mathrm{II})(Y, Z) = (\nabla_Y \mathrm{II})(X, Z) \tag{3.6}$$

for any tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where as usual

$$(\nabla_X \mathrm{II})(Y, Z) = \nabla^\perp_X (\mathrm{II}(Y, Z)) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z).$$

Observe that, for every normal field $\zeta \in \mathfrak{X}^\perp(\Sigma)$, it holds

$$\langle (\nabla_X A_\zeta) Y, Z \rangle = \langle \nabla_X \mathrm{II}(Y, Z), \zeta \rangle + \langle \mathrm{II}(Y, Z), \nabla^\perp_X \zeta \rangle,$$

where

$$(\nabla_X A_\zeta) Y = \nabla_X (A_\zeta Y) - A_\zeta(\nabla_X Y).$$

Using this into (3.6), Codazzi equation can be written equivalently as

$$(\nabla_X A_\zeta) Y = (\nabla_Y A_\zeta) X + A_{\nabla^\perp_X \zeta} Y - A_{\nabla^\perp_Y \zeta} X \tag{3.7}$$

for any tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$ and normal vector field $\zeta \in \mathfrak{X}^\perp(\Sigma)$. Finally, in our case the Ricci equation of Σ is given by

$$\langle R^\perp(X, Y)\zeta_1, \zeta_2 \rangle = -\langle [A_{\zeta_1}, A_{\zeta_2}] X, Y \rangle, \tag{3.8}$$

for any tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$ and normal vector fields $\zeta_1, \zeta_2 \in \mathfrak{X}^\perp(\Sigma)$. Here, $R^\perp$ denotes the normal curvature,

$$R^\perp(X, Y)\zeta = \nabla^\perp_{[X,Y]}\zeta - [\nabla^\perp_X, \nabla^\perp_Y]\zeta,$$

and $[A_{\zeta_1}, A_{\zeta_2}] = A_{\zeta_1} \circ A_{\zeta_2} - A_{\zeta_2} \circ A_{\zeta_1}$.

3.2 Basic equations for submanifolds through the light cone

We start this section defining the light cone of $\mathbb{L}^{n+2}$. It will be one of the null hypersurfaces where our codimension two spacelike submanifold will be contained, and the deepest studied in this chapter. Here, we will work with an explicit globally defined normal null frame $\{\xi, \eta\}$.

Definition 3.2. The light cone Λ of $\mathbb{L}^{n+2}$ is the subset of all the non zero points of $\mathbb{L}^{n+2}$ whose norm equals 0, that is,

$$\Lambda = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x \neq \mathbf{0}\}.$$

In other words, Λ is formed by all the null vectors of $\mathbb{L}^{n+2}$.

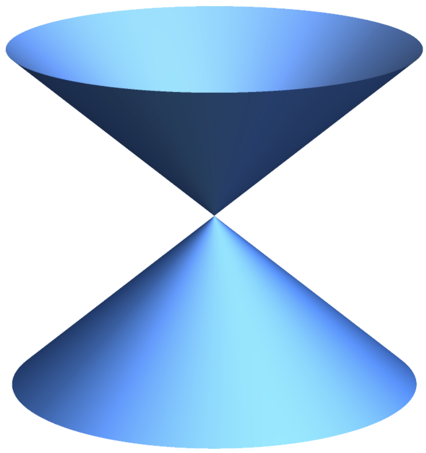

Figure 3.1: Light cone of $\mathbb{L}^{n+2}$

It corresponds to the subset of all points of the Lorentz-Minkowski spacetime which can be reached from $\mathbf{0} \in \mathbb{L}^{n+2}$ through a null (or lightlike) geodesic starting at $\mathbf{0} \in \mathbb{L}^{n+2}$. The light cone has two connected components and, since our submanifolds will be always connected, they will be contained in one of them. Without loss of generality we will work in the future component of Λ.

Definition 3.3. The future component Λ^+ of the light cone $\Lambda \subset \mathbb{L}^{n+2}$ is the subset

$$\Lambda^+ = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x_1 > 0\}.$$

Respectively, the past component Λ^- of the light cone $\Lambda \subset \mathbb{L}^{n+2}$ is the subset

$$\Lambda^- = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x_1 < 0\}.$$

Remark 3.4. Observe that the light cone defined above is actually the light cone with vertex at $\mathbf{0} \in \mathbb{L}^{n+2}$. In fact, we can define a light cone with vertex at each $\mathbf{a} \in \mathbb{L}^{n+2}$ as the subset given by $\{x \in \mathbb{L}^{n+2} : \langle x - \mathbf{a}, x - \mathbf{a} \rangle = 0, x \neq \mathbf{a}\}$. However, without loss of generality and as it is usual in the literature, we will work with Λ, the light cone centered at the origin.

Let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold and assume that $\psi(\Sigma)$ factorizes through the future connected component of the light cone. That is,

$$\langle \psi, \psi \rangle = 0 \quad \text{and} \quad \langle \psi, \mathbf{e}_1 \rangle < 0.$$

In this case

$$\xi = \psi$$

is a null vector field which is normal to the submanifold and future-pointing; it can therefore be chosen as the first vector field of our globally defined future-pointing normal null frame. We consider the positive function $u : \Sigma \to (0, +\infty)$ given by $u = -\langle \psi, \mathbf{e}_1 \rangle > 0$. It follows that

$$\nabla u = -\mathbf{e}_1^{\top},$$

where we are denoting

$$\mathbf{e}_1 = \mathbf{e}_1^{\top} + \mathbf{e}_1^{\perp}$$

as in (3.1). Thus, we get the expression

$$\mathbf{e}_1 = \mathbf{e}_1^{\perp} - \nabla u. \tag{3.9}$$

From this we have

$$\nu = \frac{1}{\sqrt{1 + \|\nabla u\|^2}} (\mathbf{e}_1 + \nabla u)$$

and

$$\langle \xi, \nu \rangle = -\frac{u}{\sqrt{1 + \|\nabla u\|^2}} < 0.$$

Therefore, by equation (2.16) we get

$$\eta = -\frac{1 + \|\nabla u\|^2}{2u^2} \psi + \frac{1}{u} (\mathbf{e}_1 + \nabla u) = -\frac{1 + \|\nabla u\|^2}{2u^2} \xi + \frac{1}{u} \mathbf{e}_1^{\perp}. \tag{3.10}$$

Thus, we have the following basic result, which is nothing but the natural extension to the n-dimensional case of Lemma 3.2 in [45], also given in Lemma 4.1 of [44] (observe that in our convention ξ and η are both future-pointing with $\langle \xi, \eta \rangle = -1$).

Proposition 3.5. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of the light cone Λ^+. Then,

$$\xi = \psi \qquad \text{and} \qquad \eta = -\frac{1 + \|\nabla u\|^2}{2u^2} \xi + \frac{1}{u} \mathbf{e}_1^{\perp} \tag{3.11}$$

are two globally defined normal null vector fields along the submanifold which are future-pointing and satisfy $\langle \xi, \eta \rangle = -1$.

Next we compute the associated shape operators, obtaining the respective n-dimensional version of Proposition 3.4 in [45], also given in Proposition 4.3 of [44].

Proposition 3.6. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of the light cone Λ^+. Then, the shape operators associated to the normal null frame defined in (3.11) are given by

$$A_\xi = I \qquad \text{and} \qquad A_\eta = -\frac{1 + \|\nabla u\|^2}{2u^2} I + \frac{1}{u} \nabla^2 u. \tag{3.12}$$

In particular, the corresponding null mean curvatures are

$$\theta_\xi = 1 \quad \text{and} \quad \theta_\eta = \frac{2u\Delta u - n(1 + \|\nabla u\|^2)}{2nu^2}, \tag{3.13}$$

where recall that ∇^2 and Δ stands respectively for the Hessian and the Laplacian operators defined in Section 2.1.

Proof. Taking into account Weingarten formula (2.5), we get

$$\overline{\nabla}_X \xi = X = A_\xi X + \nabla^\perp_X \xi$$

for every $X \in \mathfrak{X}(\Sigma)$, so that $A_\xi = I$ and $\nabla^\perp_X \xi = 0$. To obtain the expression of A_η we observe from (3.10) that

$$A_\eta = -\frac{1 + \|\nabla u\|^2}{2u^2} A_\xi + \frac{1}{u} A_{\mathbf{e}_1^\perp}.$$

Using (3.9), for every $X \in \mathfrak{X}(\Sigma)$

$$0 = \overline{\nabla}_X \mathbf{e}_1 = \overline{\nabla}_X \mathbf{e}_1^\perp - \overline{\nabla}_X \nabla u. \tag{3.14}$$

On the other hand we compute

$$\overline{\nabla}_X \mathbf{e}_1^\perp = A_{\mathbf{e}_1^\perp} X + \nabla^\perp_X \mathbf{e}_1^\perp$$

and

$$\overline{\nabla}_X \nabla u = \nabla_X \nabla u - \mathrm{II}(\nabla u, X),$$

where we have used the Gauss and Weingarten formulas (2.4) and (2.5). Inserting this into (3.14) we obtain

$$0 = A_{\mathbf{e}_1^\perp} X + \nabla_X^\perp \mathbf{e}_1^\perp - \nabla_X \nabla u + \mathrm{II}(\nabla u, X)$$

and from here we get

$$A_{\mathbf{e}_1^\perp} X = \nabla_X \nabla u. \tag{3.15}$$

Thus, we have

$$A_\eta = -\frac{1 + \|\nabla u\|^2}{2u^2} I + \frac{1}{u} \nabla^2 u$$

as we wanted to prove. Finally, tracing the expressions for A_ξ and A_η we obtain (3.13). □

Remark 3.7. Observe that since $\langle \eta, \eta \rangle = 0$ we have $\langle \nabla_X^\perp \eta, \eta \rangle = 0$ and from $\langle \xi, \eta \rangle = -1$ we also infer $\langle \nabla_X^\perp \eta, \xi \rangle = 0$. Therefore $\nabla_X^\perp \eta = 0$. Since we already know that $\nabla_X^\perp \xi = 0$, the global null frame $\{\xi, \eta\}$ is parallel in the normal bundle and, in particular, the normal connection is flat. This was already observed in Remark 4.2 (b) of [44].

On the other hand, using Proposition 3.6 and formulas (3.4) and (3.5), we easily see that the Ricci and scalar curvatures of Σ are given by

$$\begin{aligned} \mathrm{Ric}(X, Y) = &(n-1)\langle \mathbf{H}, \mathbf{H} \rangle \langle X, Y \rangle \\ &+ \frac{(n-2)}{nu} (\Delta u \langle X, Y \rangle - n \mathrm{Hess}\, u(X, Y)) \end{aligned} \tag{3.16}$$

and

$$\mathrm{Scal} = n(n-1)\langle \mathbf{H}, \mathbf{H} \rangle. \tag{3.17}$$

Finally, since $\nabla^\perp \xi = \nabla^\perp \eta = 0$ and $A_\xi = I$, Codazzi equation (3.7) reduces to

$$(\nabla_X A_\eta) Y = (\nabla_Y A_\eta) X \tag{3.18}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, and Ricci equation holds trivially since $R^\perp \equiv 0$ and $[A_\xi, A_\eta] = 0$.

3.3 Totally umbilical submanifolds through the light cone

In this section, and as a first application of our approach, we derive a classification of the codimension two totally umbilical spacelike submanifolds which factorize through the light cone Λ^+ of $\mathbb{L}^{n+2}$. Recall that an n-dimensional submanifold Σ is said to be totally umbilical if it is umbilical with respect to all possible normal directions $\zeta \in \mathfrak{X}^\perp(\Sigma)$ (see Definition 2.17 in Subsection 2.3.4).

Then, let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of the light cone Λ^+ of $\mathbb{L}^{n+2}$ and consider $\{\xi, \eta\}$ the globally defined future-pointing normal null frame given in Proposition 3.5. It follows from Proposition 3.6 that $A_\xi = I$, so that Σ is totally umbilical if, and only if, it is umbilical with respect to the normal direction η.

Below we describe the following example of codimension two totally umbilical spacelike submanifolds through Λ^+.

Example 3.8. Let $\mathbf{a} \in \mathbb{L}^{n+2}$ such that $\mathbf{a} \neq 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = c$ with $c \in \{-1, 0, 1\}$. We define

$$\Sigma(\mathbf{a}, \tau) = \{p \in \Lambda^+ : \langle p, \mathbf{a} \rangle = \tau\} \tag{3.19}$$

for a certain $\tau \in \mathbb{R}$, $\tau > 0$. If we consider

$$\begin{aligned} F_{\mathbf{a}} : \mathbb{L}^{n+2} &\to \mathbb{R}^2 \\ x &\mapsto (\langle x, x \rangle, \langle x, \mathbf{a} \rangle) \end{aligned}$$

we can see $\Sigma(\mathbf{a}, \tau)$ as $\Sigma(\mathbf{a}, \tau) = F_{\mathbf{a}}^{-1}(0, \tau)$. It is easy to see that

$$d(F_{\mathbf{a}})_x(\mathbf{v}) = (2\langle \mathbf{v}, x \rangle, \langle \mathbf{v}, \mathbf{a} \rangle)$$

for every $x \in \mathbb{L}^{n+2}$ and for every $\mathbf{v} \in T_x\mathbb{L}^{n+2} = \mathbb{L}^{n+2}$. From here, it follows that $d(F_{\mathbf{a}})_x$ is onto if, and only if, x and $\mathbf{a}$ are linearly independent. In particular, $(0, \tau)$ is a regular value of $F_{\mathbf{a}}$ if, and only if, $\Sigma(\mathbf{a}, \tau) \neq \emptyset$ and p and $\mathbf{a}$ are linearly independent for every $p \in \Sigma(\mathbf{a}, \tau)$. A detailed analysis of this condition shows that this is true for every $\tau > 0$ and $c \in \{-1, 0, 1\}$. In all those cases, $\Sigma(\mathbf{a}, \tau)$ is a codimension two spacelike submanifold which factorizes through Λ^+ and having $T_p^\perp\Sigma(\mathbf{a}, \tau) = \text{span}\{p, \mathbf{a}\}$ for every $p \in \Sigma(\mathbf{a}, \tau)$.

Let us denote $\Sigma = \Sigma(\mathbf{a}, \tau) \subset \Lambda^+$. Then, defining

$$\xi(p) = p \qquad \text{and} \qquad \eta(p) = \frac{c}{2\tau^2}p - \frac{1}{\tau}\mathbf{a}$$

we obtain a normal null frame $\{\xi, \eta\}$ such that $\langle \xi, \eta \rangle = -1$. Observe here that, for every $X \in \mathfrak{X}(\Sigma)$

$$\overline{\nabla}_X \xi = X,$$

which implies

$$A_\xi = I.$$

On the other hand $\overline{\nabla}\mathbf{a} = 0$, so

$$\overline{\nabla}_X \eta = \frac{c}{2\tau^2} X$$

for every $X \in \mathfrak{X}(\Sigma)$, and

$$A_\eta = \frac{c}{2\tau^2} I.$$

As a consequence, $\Sigma = \Sigma(\mathbf{a}, \tau)$ is a totally umbilical submanifold of $\mathbb{L}^{n+2}$ which factorizes through the light cone Λ^+. We also see that, if $\mathbf{a}$ is null, then η is a totally geodesic normal direction.

At this point we can know more about the geometry of this kind of submanifolds. As a first step, using (2.24), we compute its mean curvature vector field

$$\mathbf{H} = -\frac{c}{\tau^2} p + \frac{1}{\tau} \mathbf{a},$$

and we get the following expression for its norm

$$\langle \mathbf{H}, \mathbf{H} \rangle = -\frac{c}{\tau^2}.$$

This implies, for instance, that $\Sigma(\mathbf{a}, \tau)$ is marginally trapped if, and only if, $\mathbf{a}$ is a null vector. In this way, for every $X, Y \in \mathfrak{X}(\Sigma(\mathbf{a}, \tau))$ we also obtain, from (3.4) and (3.5), the expressions for the Ricci tensor and the scalar curvature,

$$\mathrm{Ric}(X, Y) = -\frac{c}{\tau^2}(n-1)\langle X, Y \rangle,$$

and

$$\mathrm{Scal} = -\frac{c}{\tau^2} n(n-1).$$

Therefore, we can distinguish:

(i) If $c = 1$ ($\mathbf{a}$ is spacelike), $\mathrm{Ric}(X, Y) = -\frac{n-1}{\tau^2}\langle X, Y \rangle$ and $\mathrm{Scal} = -\frac{n(n-1)}{\tau^2}$. Thus, $\Sigma(\mathbf{a}, \tau)$ is isometric to the hyperbolic space $\mathbb{H}^n(-\tau)$ with constant sectional curvature $-1/\tau^2$.

(ii) If $c = 0$ ($\mathbf{a}$ is null), $\mathrm{Ric}(X, Y) = \mathrm{Scal} = 0$, so that $\Sigma(\mathbf{a}, \tau)$ is isometric to the flat Euclidean space $\mathbb{R}^n$.

(iii) If $c = -1$ ($\mathbf{a}$ is timelike), $\mathrm{Ric}(X, Y) = \frac{n-1}{\tau^2}\langle X, Y \rangle$ and $\mathrm{Scal} = \frac{n(n-1)}{\tau^2}$. This implies that $\Sigma(\mathbf{a}, \tau)$ is isometric to the Euclidean sphere $\mathbb{S}^n(\tau)$ with constant sectional curvature $1/\tau^2$. Moreover, notice that this is the only compact case.

Our next result characterizes $\Sigma(\mathbf{a},\tau)$ in the previous example as the only codimension two totally umbilical spacelike submanifolds through Λ^+.

Theorem 3.9. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two totally umbilical spacelike submanifold which factorizes through Λ^+. Then there exists $\mathbf{a} \in \mathbb{L}^{n+2}$, $\mathbf{a} \neq 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = c \in \{-1, 0, 1\}$, and there exists $\tau \in \mathbb{R}$, $\tau > 0$, such that

$$\psi(\Sigma) \subset \Sigma(\mathbf{a}, \tau).$$

Corollary 3.10. The only complete codimension two totally umbilical spacelike submanifolds which factorize through Λ^+ are the submanifolds

$$\Sigma(\mathbf{a},\tau) = \{p \in \Lambda^+ : \langle p, \mathbf{a} \rangle = \tau\}$$

with $\mathbf{a} \in \mathbb{L}^{n+2}$, $\mathbf{a} \neq 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = c \in \{-1, 0, 1\}$, and $\tau \in \mathbb{R}$, $\tau > 0$. In particular, the only compact ones are the submanifolds $\Sigma(\mathbf{a},\tau)$ with $\langle \mathbf{a}, \mathbf{a} \rangle = -1$.

Proof. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two totally umbilical spacelike submanifold which factorizes through Λ^+ and consider $\{\xi, \eta\}$ the global normal null frame along the submanifold given in Proposition 3.5 . We know that Σ is totally umbilical if, and only if, η is an umbilical direction, that is if, and only if,

$$A_\eta = \lambda I$$

for some function $\lambda \in \mathcal{C}^\infty(\Sigma)$.

Observe that, if we consider

$$\begin{aligned} \nabla A_\eta : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) &\to \mathfrak{X}(\Sigma) \\ (X, Y) &\mapsto (\nabla A_\eta)(X, Y) = (\nabla_X A_\eta) Y, \end{aligned}$$

equation (3.18) is equivalent to the symmetry of ∇A_η, that is,

$$\nabla A_\eta(X, Y) = \nabla A_\eta(Y, X)$$

for every $X, Y \in \mathfrak{X}(\Sigma)$. Now, taking into account that η is an umbilical direction, it follows

$$\begin{aligned}(\nabla A_\eta)(X,Y) &= \nabla_Y(\lambda X) - A_\eta(\nabla_Y X)\\ &= \lambda(X)Y + \lambda\nabla_Y X - \lambda\nabla_Y X\\ &= \lambda(Y)X = (\nabla A_\eta)(Y,X)\end{aligned}$$

for every $X, Y \in \mathfrak{X}(\Sigma)$. If we choose X and Y being linearly independent, we can easily check that it has to be $X(\lambda) = 0$ for every $X \in \mathfrak{X}(\Sigma)$ and thus, λ has to be constant.

With that λ, define $Q = -\eta + \lambda\xi$. Hence, for every $X \in \mathfrak{X}(\Sigma)$

$$\overline{\nabla}_X Q = -\overline{\nabla}_X \eta + \lambda\overline{\nabla}_X \xi = -\lambda X + \lambda X = 0, \tag{3.20}$$

which implies that $Q \in \mathbb{L}^{n+2}$ is a constant vector, $Q \neq 0$.

Observe that it also holds

$$\langle Q, Q\rangle = 2\lambda, \quad \langle Q, \xi\rangle = 1, \text{ and } \quad \langle Q, \eta\rangle = -\lambda.$$

If $\lambda \neq 0$, let $\tau = 1/\sqrt{2|\lambda|} > 0$ and $\mathbf{a} = \tau Q$. Therefore, $\langle \mathbf{a}, \mathbf{a}\rangle = c = \pm 1$ and $\langle \psi, \mathbf{a}\rangle = \tau > 0$, which means that $\psi(\Sigma) \subset \Sigma(\mathbf{a}, \tau)$. If $\lambda = 0$, let $\mathbf{a} = Q$; hence $\langle \mathbf{a}, \mathbf{a}\rangle = c = 0$ and $\langle \psi, \mathbf{a}\rangle = \tau = 1$, so that $\psi(\Sigma) \subset \Sigma(\mathbf{a}, \tau)$, which finishes the proof. □

3.4 Compactness of submanifolds through the light cone

As observed in Proposition 5.1 of [44] (see also Proposition 5.1 and Remark 5.2 of [45]), if Σ is compact then it is a topological n-sphere. In this section we go further by showing that Σ is in fact conformally diffeomorphic to the Euclidean sphere and also giving a compactness criterion under an appropriate bound on the growth of the function u.

First of all, we state the following technical lemma which is essentially Lemma 5.2 in [10].

Lemma 3.11. Let g be a complete metric on a Riemannian manifold Σ and let r denote the Riemannian distance function from a fixed origin $o \in \Sigma$. If a function w satisfies

$$w^{2/(n-2)}(p) \geq \frac{C}{r(p)\log(r(p))}, \quad r(p) \gg 1, \tag{3.21}$$

C a positive constant, then the conformal metric $\widetilde{g} = w^{4/(n-2)}g$ is also complete.

In fact, Lemma 5.2 in [10] is stated under the stronger hypothesis

$$w^{2/(n-2)}(p) \geq \frac{C}{r(p)}, \quad r(p) \gg 1,$$

but a detailed reading of the proof shows that the result holds true under the weaker hypothesis (3.21). The proof is the same as in [10], just replacing (5.5) in [10] by

$$L(\gamma; \alpha, \beta) \geq C_1 \left[\log(\log(r(\gamma(\beta)))) - \log(\log(r(\gamma(\alpha))))\right]. \tag{3.22}$$

For the sake of completeness and for the reader's convenience, we include below a detailed proof.

Proof. Let $A = \{p \in \Sigma : r(p) < 1\}$ and suppose $\gamma : [0, b) \to \Sigma$ is a geodesic for $\widetilde{g}$ with $\gamma(0) = o$ and which is not extendible to b. Since (Σ, g) is complete, γ cannot remain in any compact subset of Σ. In particular, with respect to A, there are two possibilities:

(i) γ leaves A in a finite time and does not return, or

(ii) γ returns to A infinitely many times.

In both cases we are interested in the length of γ outside of A, so suppose we have $0 < \alpha < \beta < b$ with $\gamma_{|_{[\alpha,\beta]}}(t) \notin A$.

Consider the partition $\alpha < t_1 < t_2 < \ldots < t_N < \beta$ such that there exists a geodesic ball $B(\gamma(t_j), R_j)$ with center $\gamma(t_j)$ and radius R_j for which

$$\gamma(t_{j+1}) \in B(\gamma(t_j), R_j),$$

and that in $B(\gamma(t_j), R_j)$ there is a chart and coordinates in which we may write the metric g as

$$g = (dr_j)^2 + r_j^2 g_\theta, \tag{3.23}$$

r_j being the geodesic distance from $\gamma(t_j)$.

For $t_{j-1} < t < t_{j+1}$ and $h > 0$ small enough we have

$$r(\gamma(t+h)) \leq r(\gamma(t)) + r_j(\gamma(t+h)) - r_j(\gamma(t)),$$

which implies

$$\frac{dr(\gamma(t))}{dt} \leq \frac{dr_j(\gamma(t))}{dt}.$$

The length in $\widetilde{g}$ of $\{\gamma(t) : t_j < t < t_{j+1}\}$ is given by

$$\begin{aligned} L_{t_j}^{t_{j+1}}(\gamma) &= \int_{t_j}^{t_{j+1}} (\widetilde{g}(\gamma'(t), \gamma'(t))^{\frac{1}{2}}\, dt \\ &= \int_{t_j}^{t_{j+1}} \frac{1}{u^2(\gamma(t))} (g(\gamma'(t), \gamma'(t))^{\frac{1}{2}}\, dt. \end{aligned}$$

Using (3.21) and (3.23) we obtain

$$\begin{aligned} L_{t_j}^{t_{j+1}}(\gamma) &\geq \int_{t_j}^{t_{j+1}} \frac{C_1}{r(\gamma(t))\log(r(\gamma(t)))} \frac{dr_j(\gamma(t))}{dt} \\ &= C_1 \left[\log(\log(r(\gamma(t_{j+1})))) - \log(\log(r(\gamma(t_j))))\right], \end{aligned}$$

where C_1 is a constant. Thus,

$$L_{\alpha}^{\beta}(\gamma) \geq C_1 \left[\log(\log(r(\gamma(\beta)))) - \log(\log(r(\gamma(\alpha))))\right]. \tag{3.24}$$

Since γ cannot remain in any compact set we can find a sequence $\{b_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} b_j = b \quad \text{and} \quad \lim_{j\to+\infty} r(\gamma(b_j)) = +\infty. \tag{3.25}$$

Now suppose that γ satisfies $i)$, that is, for some $a \in (0, b)$ we have $\gamma(t) \notin A$ for $a < t < b$. Then, by (3.24) we have

$$\lim_{j\to+\infty} L_a^{b_j}(\gamma) = +\infty$$

so that γ has infinite length in $\widetilde{g}$.

On the other hand, if γ satisfies $ii)$, then it returns to A infinitely many times. We can now let $\{b_j\}_{j\in\mathbb{N}}$ satisfying (3.25) and $\{a_j\}_{j\in\mathbb{N}}$ such that

$$b_{j-1} < a_j < b_j, \quad \gamma(a_j) \in \partial A \quad \text{and} \quad \gamma(t) \notin A \quad \text{for} \ \ a_j < t < b_j.$$

By (3.24) we have

$$\begin{aligned} L_{a_j}^{b_j}(\gamma) &\geq C_1 \left[\log(\log(r(\gamma(b_j)))) - \log(\log(r(\gamma(a_j))))\right] \\ &\geq C_1 \left[\log(\log(r(\gamma(b_j)))) - \log(\log(d_0))\right], \end{aligned}$$

where $d_0 = \max\{r(p) : p \in \partial A\}$. Summing over j we find that γ has infinite length in $\widetilde{g}$. Thus, $\widetilde{g}$ is complete. $\square$

Now we are ready to state the following result.

Proposition 3.12. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through Λ^+. Assume that Σ is complete and that the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ satisfies

$$u(p) \leq C\, r(p) \log(r(p)), \quad r(p) \gg 1, \tag{3.26}$$

where C is a positive constant and r denotes the Riemannian distance function from a fixed origin $o \in \Sigma$. Then Σ is compact and conformally diffeomorphic to the sphere $\mathbb{S}^n$. In particular, this holds if $\sup_\Sigma u < +\infty$ and, more generally, if $\limsup_{r\to+\infty} \dfrac{u}{r\log(r)} < +\infty$.

Remark 3.13. The upper bound (3.26) on the growth of u is sharp as shown later by the existence of complete and non-compact examples with $u(p) = r^2(p)$ in Example 3.19.

Proof. Observe that, for every $p \in \Sigma$, $\psi(p) = (u(p), \psi_2(p), \ldots, \psi_{n+2}(p))$, with

$$\sum_{i=2}^{n+2} \psi_i^2(p) = u^2(p) > 0. \tag{3.27}$$

Define the function $\Psi : \Sigma^n \to \mathbb{S}^n$ by

$$\Psi(p) = \frac{1}{u(p)}(\psi_2(p), \ldots, \psi_{n+2}(p)).$$

For every $p \in \Sigma$ and $\mathbf{v} \in T_p\Sigma$ we have

$$d\Psi_p(\mathbf{v}) = -\frac{\mathbf{v}(u)}{u^2(p)}(\psi_2(p), \ldots, \psi_{n+2}(p)) + \frac{1}{u(p)}(\mathbf{v}(\psi_2), \ldots, \mathbf{v}(\psi_{n+2})).$$

Denote by $\langle , \rangle_0$ the standard metric of the round sphere $\mathbb{S}^n$. Therefore, for every

$\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we have

$$\begin{aligned} \langle d\Psi_p(\mathbf{v}), d\Psi_p(\mathbf{w})\rangle_0 &= \frac{\mathbf{v}(u)\mathbf{w}(u)}{u^4(p)} \sum_{i=2}^{n+2} \psi_i^2(p) + \frac{1}{u^2(p)} \sum_{i=2}^{n+2} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) \\ &\quad - \frac{\mathbf{v}(u)}{2u^3(p)} \mathbf{w}\left(\sum_{i=2}^{n+2} \psi_i^2\right) - \frac{\mathbf{w}(u)}{2u^3(p)} \mathbf{v}\left(\sum_{i=2}^{n+2} \psi_i^2\right) \\ &= \frac{1}{u^2(p)} \left(-\mathbf{v}(u)\mathbf{w}(u) + \sum_{i=2}^{n+2} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i)\right) \\ &= \frac{1}{u^2(p)} \langle d\psi_p(\mathbf{v}), d\psi_p(\mathbf{w})\rangle = \frac{1}{u^2(p)} \langle \mathbf{v}, \mathbf{w}\rangle. \end{aligned}$$

That is,

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2} \langle , \rangle, \tag{3.28}$$

where we recall that by $\langle , \rangle$ we denote the Riemannian metric on Σ induced by the immersion ψ.

From (3.28) it follows that Ψ is a local diffeomorphism. Assume now that Σ is complete (that is, $\langle , \rangle$ is a complete Riemannian metric on Σ) and u satisfies condition (3.26). Therefore, by Lemma 3.11 applied to the function $w = u^{-(n-2)/2}$, we know that the conformal metric

$$\widetilde{\langle , \rangle} = \frac{1}{u^2} \langle , \rangle$$

is also complete on Σ. Then, equation (3.28) means that the map

$$\Psi : (\Sigma^n, \widetilde{\langle , \rangle}) \to (\mathbb{S}^n, \langle , \rangle_0)$$

is a local isometry between complete Riemannian manifolds.

Now we recall that every local isometry between complete (connected) Riemannian manifolds is a covering map (see, for instance, [14] or [32, Chapter VIII, Lemma 8.1]). Hence, Ψ is a covering map, but $\mathbb{S}^n$ being simply connected this means that Ψ is in fact a global diffeomorphism between Σ and $\mathbb{S}^n$. □

In the following example we observe that for each positive smooth function $f \colon \mathbb{S}^n \to (0, +\infty)$ we can construct an embedding $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$.

Example 3.14. Let $f : \mathbb{S}^n \to (0, +\infty)$ be a positive smooth function and set

$$\begin{aligned}\psi_f\colon \mathbb{S}^n &\to \Lambda^+ \subset \mathbb{L}^{n+2}\\ p &\mapsto (f(p), f(p)p).\end{aligned}$$

Clearly, for every $p \in \mathbb{S}^n$, $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^n$ we have

$$d(\psi_f)_p(\mathbf{v}) = (\mathbf{v}(f), \mathbf{v}(f)p + f(p)\mathbf{v})$$

and

$$\langle d(\psi_f)_p(\mathbf{v}), d(\psi_f)_p(\mathbf{w})\rangle = f^2(p)\langle \mathbf{v}, \mathbf{w}\rangle_0.$$

That is,

$$\psi_f^*(\langle , \rangle) = f^2\langle , \rangle_0, \tag{3.29}$$

what means that ψ_f determines a spacelike immersion of $\mathbb{S}^n$ through Λ^+ whose induced metric is conformal to the standard metric of the round sphere.

In this case $u = f$ and, from equation (3.12) in Proposition 3.6, we can explicitly write the second fundamental form of ψ_f in terms of the function f and the gradient and the Hessian of f with respect to the round metric $\langle , \rangle_0$. To see it first observe that, obviously, $A_\xi = I$ and $\theta_\xi = 1$. To compute A_η, let us denote by $\|\cdot\|_0^2$, ∇^0 and Hess_0 the norm, the gradient and the Hessian operator (as a symmetric (0,2) tensor) on $\mathbb{S}^n$ with respect to the standard metric $\langle , \rangle_0$. Using relation (2.11), we have

$$\|\nabla f\|^2 = \frac{1}{f^2}\|\nabla^0 f\|_0^2 \tag{3.30}$$

and

$$\frac{1+\|\nabla f\|^2}{2f^2} = \frac{f^2+\|\nabla^0 f\|_0^2}{2f^4}. \tag{3.31}$$

On the other hand, by (2.12) we also get, for every tangent vector fields $X, Y \in \mathfrak{X}(\mathbb{S}^n)$,

$$\begin{aligned}\mathrm{Hess}\, f(X,Y) &= f^2\langle \nabla_X \nabla f, Y\rangle_0\\ &= \langle \nabla^0_X \nabla^0 f, Y\rangle_0 - \frac{2}{f}\langle X, \nabla^0 f\rangle_0 \langle Y, \nabla^0 f\rangle_0\\ &\quad + \frac{1}{f}\|\nabla^0 f\|_0^2\langle X, Y\rangle_0,\end{aligned}$$

which gives

$$\nabla_X \nabla f = \frac{1}{f^2}\nabla^0_X\nabla^0 f - \frac{2}{f^3}\langle X, \nabla^0 f\rangle_0 \nabla^0 f + \frac{1}{f^3}\|\nabla^0 f\|_0^2 X \tag{3.32}$$

for every tangent vector field $X \in \mathfrak{X}(\mathbb{S}^n)$. Therefore, using (3.31) and (3.32) in (3.12) we conclude after some computations that

$$A_\eta(X) = \frac{1}{f^3}\nabla^0_X\nabla^0 f - \frac{2}{f^4}\langle X, \nabla^0 f\rangle_0 \nabla^0 f + \frac{\|\nabla^0 f\|_0^2 - f^2}{2f^4}X \tag{3.33}$$

for every $X \in \mathfrak{X}(\mathbb{S}^n)$. Thus, tracing (3.33) with respect to $\langle , \rangle_0$ we have

$$\theta_\eta = \frac{2f\Delta_0 f + (n-4)\|\nabla^0 f\|_0^2 - nf^2}{2nf^4}. \tag{3.34}$$

In the next result, and as a consequence of Proposition 3.12, we observe that every codimension two compact spacelike submanifold which factorizes through Λ^+ is, up to a conformal diffeomorphism, as in Example 3.14.

Corollary 3.15. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two compact spacelike submanifold which factorizes through Λ^+. Then there exists a conformal diffeomorphism $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ such that

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle$$

with $u = -\langle \psi, \mathbf{e}_1 \rangle = \psi_1 > 0$, and $\psi = \psi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$ and $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ is the embedding

$$\psi_f(p) = (f(p), f(p)p).$$

$$\begin{tikzcd} \Sigma^n \arrow[r, "u"] \arrow[d, shift left, "\Psi"] & (0,+\infty) \\ \mathbb{S}^n \arrow[u, shift left, "\Phi"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \Sigma^n \arrow[r, "\psi"] \arrow[d, shift left, "\Psi"] & \Lambda^+ \subset \mathbb{L}^{n+2} \\ \mathbb{S}^n \arrow[u, shift left, "\Phi"] \arrow[ur, "\psi_f"'] & \end{tikzcd}$$

In particular, the immersion ψ is an embedding.

For the proof simply consider u and Ψ as in the proof of Proposition 3.12, and recall that in this situation $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ is a conformal diffeomorphism with

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle.$$

Let $\Phi : \mathbb{S}^n \to \Sigma^n$ be the inverse of Ψ. Then taking $f = u \circ \Phi$ one has $f \circ \Psi = u$

and

$$\psi_f \circ \Psi(p) = (f(\Psi(p)), f(\Psi(p))\Psi(p)) = (u(p), \psi_2(p), \ldots, \psi_{n+2}(p)) = \psi(p),$$

that is, $\psi = \psi_f \circ \Psi$ as we wanted to see.

3.5 Trapped submanifolds through the light cone. First results

In this part of the chapter, we will focus our research on the case of trapped submanifolds. It is known that there exists no compact weakly trapped submanifold in $\mathbb{L}^{n+2}$. In fact, in [36, Theorem 2] this is proved in the more general case of stationary spacetimes. Here, and for the convenience of the reader, we give a proof using our approach.

Proposition 3.16. There exists no codimension two compact weakly trapped submanifold in $\mathbb{L}^{n+2}$.

Proof. Let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ be an n-dimensional compact weakly trapped submanifold and consider the function $u = -\langle \psi, \mathbf{e}_1 \rangle$, whose gradient is given by $\nabla u = -\mathbf{e}_1^\top = \mathbf{e}_1^\perp - \mathbf{e}_1$. Taking derivatives here, as we did in Section 3.2 for the case where the submanifold factorizes through the light cone, we obtain again

$$A_{\mathbf{e}_1^\perp} X = \nabla_X \nabla u$$

and from here we compute

$$\Delta u = \mathrm{tr}(A_{\mathbf{e}_1^\perp}) = n\langle \mathbf{H}, \mathbf{e}_1^\perp \rangle = n\langle \mathbf{H}, \mathbf{e}_1 \rangle.$$

On the other hand, since the mean curvature vector field $\mathbf{H}$ is not spacelike, it satisfies either $\langle \mathbf{H}, \mathbf{e}_1 \rangle < 0$ or $\langle \mathbf{H}, \mathbf{e}_1 \rangle > 0$ on Σ. If we suppose that $\langle \mathbf{H}, \mathbf{e}_1 \rangle < 0$, then

$$\Delta u = -n\langle \mathbf{H}, \mathbf{e}_1 \rangle > 0.$$

Now, from the divergence theorem we have

$$\int_\Sigma \Delta u \, \mathrm{d}\Sigma = 0,$$

what implies $\Delta u \equiv 0$ and gives us a contradiction. The proof for the case $\langle \mathbf{H}, \mathbf{e}_1 \rangle > 0$ ends in a similiar way. $\square$

Furthermore, from Proposition 3.6 and equation (2.24), the mean curvature vector field of a codimension two spacelike submanifold Σ which factorizes through Λ^+ is given by

$$\mathbf{H} = -\frac{1}{2nu^2}(2u\Delta u - n(1+\|\nabla u\|^2))\xi - \eta.$$

In particular, the null expansion θ_ξ is always $\theta_\xi = 1 > 0$ and

$$\langle \mathbf{H}, \mathbf{H} \rangle = -\frac{1}{nu^2}\left(2u\Delta u - n(1+\|\nabla u\|^2)\right). \tag{3.35}$$

As a consequence of these computations we have the following.

Corollary 3.17. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of the light cone Λ^+. Let u be the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$. It is satisfied:

(i) Σ is (necessarily past) weakly trapped if, and only if, u satisfies the differential equation

$$2u\Delta u - n(1+\|\nabla u\|^2) \geq 0 \quad \text{on } \Sigma. \tag{3.36}$$

(ii) Σ is (necessarily past) marginally trapped if, and only if, u satisfies the differential equation

$$2u\Delta u - n(1+\|\nabla u\|^2) = 0 \quad \text{on } \Sigma. \tag{3.37}$$

(iii) Σ is (necessarily past) trapped if, and only if, u satisfies the differential equation

$$2u\Delta u - n(1+\|\nabla u\|^2) > 0 \quad \text{on } \Sigma. \tag{3.38}$$

Corollary 3.18. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of the light cone Λ^+. Let u be the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$. The following are equivalent:

(i) Σ is (necessarily past) marginally trapped.

(ii) u satisfies the differential equation $2u\Delta u - n(1+\|\nabla u\|^2) = 0$ on Σ.

(iii) Σ has zero scalar curvature, $\mathrm{Scal} = 0$.

3.5.1 Examples

In this subsection, we present explicit examples of weakly trapped and marginally trapped submanifolds which factorize through the future component of the light cone Λ^+. In the first one, we construct a marginally trapped submanifold.

Example 3.19. Let $\psi : \mathbb{R}^n \to \mathbb{L}^{n+2}$ be a map given by

$$\psi(p) = (\frac{\|p\|^2+1}{2}, \frac{\|p\|^2-1}{2}, p).$$

We compute

$$\langle \psi(p), \psi(p)\rangle = -\frac{(\|p\|^2+1)^2 + (\|p\|^2-1)^2}{4} + \|p\|^2 = 0$$

and we also have

$$u(p) = -\langle \psi(p), \mathbf{e}_1\rangle = \frac{\|p\|^2+1}{2} > 0.$$

Therefore, $\psi(\mathbb{R}^n)$ factorizes through the light cone Λ^+.

On the other hand, for every $p \in \mathbb{R}^n$ and $\mathbf{v}, \mathbf{w} \in T_p\mathbb{R}^n$, we obtain

$$d\psi_p(\mathbf{v}) = (\|p\|\mathbf{v}, \|p\|\mathbf{v}, \mathbf{v})$$

and hence,

$$\psi^*(\langle \mathbf{v}, \mathbf{w}\rangle) = \langle d\psi_p(\mathbf{v}), d\psi_p(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle_{\mathbb{R}^n}.$$

Thus, ψ is an isometric immersion of $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$ through $\Lambda^+ \subset \mathbb{L}^{n+2}$ and, in particular, the gradient and Laplacian operators of u are respectively

$$\nabla u(p) = \nabla^{\mathbb{R}^n} u(p) = p$$

and

$$\Delta u(p) = \Delta_{\mathbb{R}^n} u(p) = n.$$

Therefore, the function u satisfies the differential equation (3.37),

$$2u\Delta u - n(1 + \|\nabla u\|^2) = n(\|p\|^2+1) - n(\|p\|^2+1) = 0$$

and, from Corollary 3.17, ψ is a marginally trapped immersion of $\mathbb{R}^n$ through $\Lambda^+ \subset \mathbb{L}^{n+2}$.

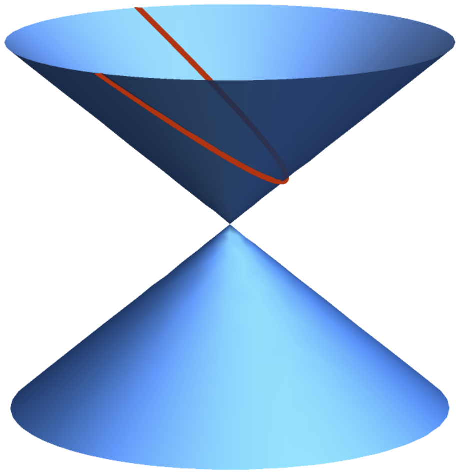

Figure 3.2: Example 3.19 with $n = 1$

In the next example we construct a weakly trapped submanifold which is marginally trapped when the dimension is $n = 2$.

Example 3.20. Let $\phi : (0, +\infty) \times \mathbb{H}^{n-1} \to \mathbb{L}^{n+2}$, $n \geq 2$, be the map given by

$$\phi(t, p) = (p, \cos(t), \sin(t)),$$

where we are denoting by $\mathbb{H}^{n-1}$ the $(n-1)$-dimensional unit hyperbolic space,

$$\mathbb{H}^{n-1} = \{p = (p_1, \dots, p_n) \in \mathbb{L}^n : \langle p, p \rangle = -1, p_1 > 0\}.$$

It is satisfied that

$$\langle \phi(t, p), \phi(t, p) \rangle = \langle p, p \rangle_{\mathbb{H}^{n-1}} + \cos(t)^2 + \sin(t)^2 = -1 + 1 = 0,$$

and

$$u(t, p) = p_1 > 0,$$

so that $\phi((0, +\infty) \times \mathbb{H}^{n-1})$ factorizes through Λ^+.

We have that for every $z = (t, p) \in (0, +\infty) \times \mathbb{H}^{n-1}$ the tangent space at z is generated by two types of vectors, $\mathbf{v}_1 = (1, \mathbf{0})$ with $\mathbf{0} \in T_p\mathbb{H}^{n-1}$ and $\mathbf{v}_2 = (0, \mathbf{w})$ with $\mathbf{w} \in T_p\mathbb{H}^{n-1}$. Thus, we compute

$$d\phi_z(1, \mathbf{0}) = (0, -\sin(t), \cos(t))$$

and

$$d\phi_z(0, \mathbf{w}) = (\mathbf{w}, 0, 0) \quad \text{for every } \mathbf{w} \in T_p\mathbb{H}^{n-1}.$$

From here, we can easily see that

$$\phi^*(\langle , \rangle) = dt^2 + \langle , \rangle_{\mathbb{H}^{n-1}},$$

that is, the induced metric is nothing but the product metric on $(0, +\infty) \times \mathbb{H}^{n-1}$. In other words, ϕ gives an isometric immersion of the Riemannian product manifold $(0, +\infty) \times \mathbb{H}^{n-1}$ through $\Lambda^+ \subset \mathbb{L}^{n+2}$.

Therefore, in this case $u(t, p) = v(p)$ for every $(t, p) \in (0, +\infty) \times \mathbb{H}^{n-1}$, where the function $v : \mathbb{H}^{n-1} \to (0, +\infty)$ is given by $v(p) = -\langle p, \mathbf{e}_1\rangle_{\mathbb{L}^n}$ for every $p \in \mathbb{H}^{n-1}$, with $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{L}^n$. In particular,

$$\nabla u(t, p) = (0, Dv(p))$$

where D denotes the gradient operator on $\mathbb{H}^{n-1}$. Since $v(p) = -\langle p, \mathbf{e}_1\rangle_{\mathbb{L}^n}$, it is clear that $Dv(p) = -\mathbf{e}_1^\top$, where $\mathbf{e}_1^\top$ denotes here the component which is tangent to $\mathbb{H}^{n-1}$ as a spacelike hypersurface of $\mathbb{L}^n$. Hence, the vector $\mathbf{e}_1$ decomposes along $\mathbb{H}^{n-1}$ as

$$\mathbf{e}_1 = -Dv(p) + v(p)p$$

for every $p \in \mathbb{H}^{n-1}$ and

$$\|Dv\|^2 = -1 + v^2.$$

Since $\nabla u = (0, Dv)$, this is equivalent to

$$\|\nabla u\|^2 = -1 + u^2.$$

On the other hand, for every $\mathbf{v} \in T_p\mathbb{H}^{n-1}$ we have

$$\nabla^0_{\mathbf{v}}\mathbf{e}_1 = 0 = -\nabla^0_{\mathbf{v}}Dv + \mathbf{v}(v(p)p) = -D_{\mathbf{v}}Dv + v(p)\mathbf{v} + \mathbf{v}(v)p$$

where ∇^0 and D denote, respectively, the Levi-Civita connections of $\mathbb{L}^n$ and $\mathbb{H}^{n-1}$. From here we have that $D_{\mathbf{v}}Dv = v(p)\mathbf{v}$ for every $\mathbf{v} \in T_p\mathbb{H}^{n-1}$, and then

$$\Delta_{\mathbb{H}^{n-1}}v = (n-1)v.$$

Hence,

$$\Delta u(t, p) = \Delta_{\mathbb{H}^{n-1}}v(p) = (n-1)u(t, p)$$

and from these computations we finally get

$$2u\Delta u - n(1 + \|\nabla u\|^2) = (n-2)u^2.$$

Taking now into account equations (3.36) and (3.37), we have that Σ is a weakly trapped submanifold which is marginally trapped if, and only if $n = 2$.

3.6 Non-existence of weakly trapped submanifolds through the light cone

We know from Proposition 3.16 that, in particular, there is no compact weakly trapped submanifold which factorizes through the light cone of $\mathbb{L}^{n+2}$. Motivated by this fact, our main goal in this section is to obtain some other conditions, without assuming the compactness of Σ, that ensure the non-existence of weakly trapped submanifolds through the light cone. The following corollary is a direct consequence of Proposition 3.12 and Proposition 3.16.

Corollary 3.21. There exists no codimension two complete weakly trapped submanifold $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ satisfies

$$u \leq Cr \log r, \qquad r >> 1.$$

In particular, there is no codimension two complete weakly trapped immersed submanifold in $\Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function u is bounded from above.

Now, we extend this non-existence result to the more general case of stochastically complete submanifolds (see Subsection 2.3.2).

Proposition 3.22. There exists no codimension two stochastically complete weakly trapped immersed submanifold factorizing through $\Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ is bounded from above.

Proof. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be an n-dimensional stochastically complete weakly trapped submanifold such that $\psi(\Sigma) \subset \Lambda^+$. If we define $u = -\langle \psi, \mathbf{e}_1 \rangle$ as usual, it satisfies (3.36) in Corollary 3.17,

$$n(1 + \|\nabla u\|^2) \leq 2u\Delta u. \tag{3.39}$$

Suppose that u is bounded from above, that is, $u^* = \sup_\Sigma u < +\infty$. Since Σ is stochastically complete, by the weak maximum principle for the Laplacian (2.9) there exists a sequence $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma$ with

$$\Delta u(p_k) < \frac{1}{k} \quad \text{for every} \quad k \in \mathbb{N}$$

and

$$\lim_{k\to+\infty} u(p_k) = u^*.$$

Putting this into (3.39) we obtain

$$n \leq n(1 + \|\nabla u(p_k)\|^2) \leq 2u(p_k)\Delta u(p_k) < 2\frac{u(p_k)}{k},$$

and making now $k \to +\infty$ we get

$$n \leq 0,$$

which is not possible. □

For the proof of the next theorem we need the following analytical result whose proof can be derived from that of Theorem 3.3 in [37].

Theorem 3.23. Let $(\Sigma, \langle , \rangle)$ be a complete Riemannian manifold and let $v \geq 0$ be a solution of

$$v\Delta v + av^2 - bv \geq -A\|\nabla v\|^2 \tag{3.40}$$

on Σ, with $a \leq 0$, $b > 0$ and $A \in \mathbb{R}$. Suppose that for some $\alpha > 1$, $\beta > -1$, $\beta \geq A$

$$v \in L^{\alpha(\beta+1)}(\Sigma). \tag{3.41}$$

Then $v \equiv 0$.

Proof. Let $v \geq 0$ be a solution of (3.40). Fix $\varepsilon > 0$ and set

$$w_\varepsilon = (v^2 + \varepsilon)^{\frac{\beta+1}{2}}.$$

We compute

$$w_\varepsilon \Delta w_\varepsilon = (\beta+1)(v^2+\varepsilon)^\beta v\Delta v + \\ (\beta+1)(v^2+\varepsilon)^\beta(1 + (\beta-1)\frac{v^2}{v^2+\varepsilon})\|\nabla v\|^2.$$

If follows from (3.40) and the assumptions $a \leq 0$ and $b > 0$ that

$$
\begin{aligned}
w_\varepsilon \Delta w_\varepsilon \geq & (\beta+1)(v^2+\varepsilon)^\beta \left(\left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon} \right) \|\nabla v\|^2 + bv - av^2 \right) \\
& \geq (\beta+1)(v^2+\varepsilon)^\beta \left(\left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon} \right) \|\nabla v\|^2 + a(1 - \frac{v^2}{v^2+\varepsilon}) \right) \\
& = (\beta+1)(v^2+\varepsilon)^\beta \left(a\varepsilon + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon} \right) \right).
\end{aligned}
$$

Let $\tilde{r}(t) \in C^1(\mathbb{R})$ and $s(t) \in C^0(\mathbb{R})$ defined as

$$\tilde{r}(t) = t^{\alpha-2} \quad \text{and} \quad s(t) = c_\alpha t^{\alpha-2},$$

where $\alpha > 1$ and $c_\alpha = \min\{\alpha - 1, 1\}$. They satisfy the conditions

$$\tilde{r}(w_\varepsilon) \geq 0 \tag{3.42}$$

and

$$\tilde{r}(w_\varepsilon) + w_\varepsilon \tilde{r}'(w_\varepsilon) = (\alpha-1) w_\varepsilon^{\alpha-2} \geq c_\alpha w_\varepsilon^{\alpha-2} = s(w_\varepsilon) > 0. \tag{3.43}$$

Consider the vector field

$$Z = w_\varepsilon \tilde{r}(w_\varepsilon) \nabla w_\varepsilon = w_\varepsilon^{\alpha-1} \nabla w_\varepsilon$$

and, for fixed t and $\delta > 0$, let ψ_δ be the Lipschitz function defined by

$$\psi_\delta(p) = \begin{cases} 1 & \text{if} \quad r(p) \leq t \\ \frac{t+\delta-r(p)}{\delta} & \text{if} \quad t < r(p) < t+\delta \\ 0 & \text{if} \quad r(p) \geq t+\delta. \end{cases}$$

Using conditions (3.42), (3.43) and the definition of ψ_δ we compute

$$
\begin{aligned}
\operatorname{div}(\psi_\delta Z) &= \psi_\delta \operatorname{div}(Z) + \langle \nabla \psi_\delta, Z \rangle \\
&= \left(w_\varepsilon^{\alpha-1} \Delta w_\varepsilon + (\alpha-1) w_\varepsilon^{\alpha-2} \|\nabla w_\varepsilon\|^2 \right) \psi_\delta - \frac{1}{\delta} \langle \nabla r, w_\varepsilon^{\alpha-1} \nabla w_\varepsilon \rangle \\
&= \left(w_\varepsilon^{\alpha-2} w_\varepsilon \Delta w_\varepsilon + (\alpha-1) w^{\alpha-2} \|\nabla w\|^2 \right) \psi_\delta - \frac{1}{\delta} \langle \nabla r, w^{\alpha-1} \nabla w \rangle \\
&\geq w_\varepsilon^{\alpha-2} (\beta+1)(v^2+\varepsilon)^\beta \left(a\varepsilon + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon} \right) \|\nabla v\|^2 \right) \psi_\delta \\
&+ c_\alpha w_\varepsilon^{\alpha-2} \|\nabla w_\varepsilon\|^2 \psi_\delta - \frac{1}{\delta} \langle \nabla r, w_\varepsilon^{\alpha-2} \nabla w_\varepsilon \rangle \chi_{B_{t+\delta} \setminus B_t},
\end{aligned}
$$

where we have used $\nabla\psi_\delta = -\frac{1}{\delta}\nabla r \chi_{B_{t+\delta}\setminus B_t}$.

Then, integrating and using the divergence theorem and Cauchy-Schwarz inequality we obtain

$$\int_{B_t} w_\varepsilon^{\alpha-2}(\beta+1)(v^2+\varepsilon)^\beta \left(a\varepsilon + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right)\psi_\delta + \int_{B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2 \le \frac{1}{\delta}\int_{B_{t+\delta}\setminus B_t} w_\varepsilon^{\alpha-1}\|\nabla w_\varepsilon\|. \tag{3.44}$$

By Hölder inequality the integral on the right-hand side is bounded above as follows

$$\begin{aligned}\int_{\bar{B}_{t+\delta}\setminus B_t} w_\varepsilon^{\alpha-1}\|\nabla w_\varepsilon\| &= \int_{\bar{B}_{t+\delta}\setminus B_t}\left(\frac{1}{\sqrt{\delta c_\alpha}} w_\varepsilon^{\alpha/2}\right)\left(\frac{\sqrt{c_\alpha}}{\sqrt{\delta}} w_\varepsilon^{\alpha/2-1}\|\nabla w_\varepsilon\|\right)\\ &\le \left(\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t}\frac{w_\varepsilon^\alpha}{c_\alpha}\right)^{\frac{1}{2}}\left(\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|\right)^{\frac{1}{2}}.\end{aligned}$$

Inserting this into inequality (3.44) and letting $\delta \to 0^+$ we obtain that

$$\int_{B_t} w_\varepsilon^{\alpha-2}(\beta+1)(v^2+\varepsilon)^\beta\left(a\varepsilon + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right) + + \int_{B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2 \le \left(\int_{\partial B_t}\frac{w_\varepsilon^\alpha}{c_\alpha}\right)^{\frac{1}{2}}\left(\int_{\partial B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2\right)^{\frac{1}{2}}, \tag{3.45}$$

where we have used the co-area formula, that is,

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t}\frac{w_\varepsilon^\alpha}{c_\alpha} = \int_{\partial B_t}\frac{w_\varepsilon^\alpha}{c_\alpha}$$

and

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2 = \int_{\partial B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2.$$

As $\varepsilon \to 0$, then $w_\varepsilon \to w_0 = v^{\beta+1}$. Therefore, using the dominated convergence theorem in (3.44) we get

$$\begin{aligned}(\beta+1)(\beta-A)\int_{B_t} v^{2\beta}w_0^{\alpha-2}\|\nabla v\|^2 + \int_{B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2 \\ \leq \left(\int_{\partial B_t}\frac{w_0^\alpha}{c_\alpha}\right)^{\frac{1}{2}}\left(\int_{\partial B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.46}$$

We define now

$$h(t) = \int_{B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2,$$

and then, by the co-area formula, h is Lipschitz and

$$h'(t) = \int_{\partial B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2.$$

From our assumptions on β and A, we know that

$$(\beta+1)(\beta-A)\int_{B_t} v^{2\beta}w_0^{\alpha-2}\|\nabla v\|^2 \geq 0$$

so, from (3.46), it is satisfied

$$h(t) \leq \left(\int_{\partial B_t}\frac{w_0^\alpha}{c_\alpha}\right)^{\frac{1}{2}} h'(t)^{\frac{1}{2}}. \tag{3.47}$$

Our aim now is to show that $w_0 = v^{\beta+1}$ is constant. Let us suppose it is not and reason by contradiction.Then, if w_0 is not constant, there exists $R_0 >> 1$ such that $h(t) > 0$ for every $t \geq R_0$. Then, dividing in (3.47) by $h(t)$ we have

$$1 \leq \frac{h'(t)}{h(t)^2}\int_{\partial B_t}\frac{w_0^\alpha}{c_\alpha}$$

or, equivalently,

$$\frac{h'(t)}{h(t)^2} \geq \left(\int_{\partial B_t}\frac{w_0^\alpha}{c_\alpha}\right)^{-1}.$$

Taking $R_0 \leq r < R$ and integrating the previous inequality, we obtain

$$\begin{aligned}\left(\int_{B_r} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2\right)^{-1} &= \frac{1}{h(r)} \geq \frac{1}{h(r)} - \frac{1}{h(R)} \\ &= \int_r^R \frac{h'(t)}{h(t)^2} \geq \int_r^R \left(\int_{\partial B_t}\frac{w_0^\alpha}{c_\alpha}\right)^{-1}.\end{aligned}$$

Since $w_0 = v^{\beta+1}$, from here it follows

$$\int_r^R \left(\int_{\partial B_t} v^{\alpha(\beta+1)}\right)^{-1} \leq C \left(\int_{B_r} w_0^{\alpha-2}\|\nabla w_0\|^2\right)^{-1} < +\infty,$$

where $C \in \mathbb{R}$. Then,

$$\left(\int_{\partial B_t} v^{\alpha(\beta+1)}\right)^{-1} \in L^1(+\infty). \tag{3.48}$$

If we define now

$$\phi(t) = \int_{B_t} v^{\alpha(\beta+1)}$$

we obtain

$$\phi'(t) = \int_{\partial B_t} v^{\alpha(\beta+1)} \geq 0.$$

Taking into account (3.48), $\phi'(t)$ has to satisfy

$$\lim_{t\to+\infty} \frac{1}{\phi'(t)} = 0,$$

or equivalently

$$\lim_{t\to+\infty} \phi(t) = +\infty.$$

As ϕ is a non-decreasing function, it implies that ϕ tends to infinity, that is,

$$\lim_{t\to+\infty} \int_{B_t} v^{\alpha(\beta+1)} = \int_{\Sigma} v^{\alpha(\beta+1)} = +\infty.$$

However, this contradicts the assumption (3.41), and we get that w_0 has to be constant and so does v. Taking into account that v satisfies equation (3.40),

$$av^2 - bv \geq 0$$

with $a \leq 0$ and $b > 0$, then we conclude that $v \equiv 0$. $\square$

As a consequence of Theorem 3.23 we have the following.

Theorem 3.24. There is no codimension two complete weakly trapped submanifold $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ satisfies

$$u \in L^q(\Sigma) \tag{3.49}$$

for any $q > 0$.

Proof. Let Σ be a codimension two complete weakly trapped submanifold through the light cone Λ^+ and assume that $u \in L^q(\Sigma)$ for some $q > 0$. Define $v = u^2 > 0$. From equation (3.36) we have

$$v\Delta v - nv \geq \frac{n+2}{4}\|\nabla v\|^2, \tag{3.50}$$

so that we can apply Theorem 3.23 with the choices $a = 0$, $b = n$, $A = -(n+2)/4$. Note that, since $n \geq 2$, then $A \leq -1$ and the only condition on β required in Theorem 3.23 is now $\beta > -1$. Choose then $\beta = -1 + q/4 > -1$ and take $\alpha = 2$, so that

$$\alpha(\beta + 1) = \frac{q}{2}.$$

This implies that

$$v \in L^{\alpha(\beta+1)}(\Sigma) \tag{3.51}$$

and, by Theorem 3.23, $v \equiv 0$, which is a contradiction, completing the proof of the theorem. □

3.7 Codimension two spacelike submanifolds through a null hyperplane

To finish this chapter we briefly study the case of another null hypersurface of the Lorentz-Minkowski spacetime: a null hyperplane. Let us start by giving the definition of such a hypersurface.

Definition 3.25. Let $\mathbf{a} \in \mathbb{L}^{n+2}$ be a null vector. The subset

$$\mathcal{L}_{\mathbf{a}} = \{x \in \mathbb{L}^{n+2} : \langle x, \mathbf{a}\rangle = 0, x \neq \mathbf{a}\}$$

is a null hyperplane on the Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$.

Let us suppose $\mathbf{a}$ future-pointing and let $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the null hyperplane $\mathcal{L}_{\mathbf{a}}$. In this case

$$\xi = \mathbf{a}$$

is a future-pointing null vector field which is normal to the submanifold and hence, it can be chosen as the first vector field of our globally defined future-pointing normal null frame. We define the function $u : \Sigma \to \mathbb{R}$ by $u = -\langle \psi, \mathbf{e}_1 \rangle = \psi_1$. Following the expressions obtained in (3.2) and (3.3) we have

$$\nu = \frac{\mathbf{e}_1 + \nabla u}{\sqrt{1 + \|\nabla u\|^2}},$$

and

$$\eta = -\frac{1 + \|\nabla u\|^2}{2\langle \mathbf{e}_1, \mathbf{a} \rangle^2}\mathbf{a} - \frac{1}{\langle \mathbf{e}_1, \mathbf{a} \rangle}(\mathbf{e}_1 + \nabla u).$$

Therefore, we have that η is a globally defined normal null vector field which is future-pointing and satisfies $\langle \xi, \eta \rangle = -1$. In this setting we can state the following.

Proposition 3.26. Let $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the null hyperplane $\mathcal{L}_{\mathbf{a}}$. Then,

$$\xi = \mathbf{a} \qquad \text{and} \qquad \eta = -\frac{1 + \|\nabla u\|^2}{2\langle \mathbf{e}_1, \mathbf{a} \rangle^2}\mathbf{a} - \frac{1}{\langle \mathbf{e}_1, \mathbf{a} \rangle}(\mathbf{e}_1 + \nabla u)$$

are two globally defined normal null vector fields along the submanifold which are future-pointing and satisfy $\langle \xi, \eta \rangle = -1$.

In the same way that in Proposition 3.6 we can compute the shape operators and null mean curvatures of Σ with respect to $\{\xi, \eta\}$.

Proposition 3.27. Let $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the null hyperplane $\mathcal{L}_{\mathbf{a}}$. Then, the shape operators associated to ξ and η are, respectively,

$$A_\xi = 0 \qquad \text{and} \qquad A_\eta = -\frac{1}{\langle \mathbf{e}_1, \mathbf{a} \rangle}\nabla^2 u$$

In particular,

$$\theta_\xi = 0 \qquad \text{and} \qquad \theta_\eta = -\frac{1}{n\langle \mathbf{e}_1, \mathbf{a} \rangle}\Delta u. \tag{3.52}$$

From the previous proposition and formula (2.24) we have the expression for the mean curvature vector field,

$$\mathbf{H} = \frac{\Delta u}{n\langle \mathbf{e}_1, \mathbf{a} \rangle}\mathbf{a} \tag{3.53}$$

and then, $\langle \mathbf{H}, \mathbf{H} \rangle = 0$. Thus, we have the following.

Proposition 3.28. Let $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the null hyperplane $\mathcal{L}_{\mathbf{a}}$. Then Σ is marginally trapped except at points where $\Delta u = 0$ on Σ.

In what follows, and without loss of generality, we may assume that the future-pointing null vector is $\mathbf{a} = (1, 0, \dots, 0, 1)$. Now we denote the null hyperplane $\mathcal{L}_{\mathbf{a}}$ simply by $\mathcal{L}$. Our next result corresponds to Proposition 3.12.

Proposition 3.29. Let $\psi : \Sigma^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which factorizes through the null hyperplane $\mathcal{L}$. Assume that Σ is complete. Then Σ is isometric to the Euclidean space $(\mathbb{R}^n, \langle, \rangle_{\mathbb{R}^n})$.

Proof. Since $\psi(\Sigma) \subset \mathcal{L}$, for every $p \in \Sigma$ we can write

$$\psi(p) = (u(p), \psi_2(p), \dots, \psi_{n+1}(p), u(p)).$$

We define the function

$$\begin{aligned} \Psi \colon \Sigma^n &\to \mathbb{R}^n \\ p &\mapsto (\psi_2(p), \dots \psi_{n+1}(p)). \end{aligned}$$

and, for every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we compute

$$d\Psi_p(\mathbf{v}) = (\mathbf{v}(\psi_2), \dots, \mathbf{v}(\psi_{n+1}))$$

and

$$\begin{aligned} \langle d\Psi_p(\mathbf{v}), d\Psi_p(\mathbf{w}) \rangle_{\mathbb{R}^n} &= \sum_{i=2}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) \\ &= -\mathbf{v}(u)\mathbf{w}(u) + \sum_{i=2}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) + \mathbf{v}(u)\mathbf{w}(u) \\ &= \langle d\psi_p(\mathbf{v}), d\psi_p(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle. \end{aligned}$$

In other words, $\Psi^*(\langle, \rangle_{\mathbb{R}^n}) = \langle, \rangle$, which means that Ψ is a local isometry. After this point, since Σ is complete and $\mathbb{R}^n$ is simply connected, we obtain that Ψ is in fact a global isometry. $\square$

In next example we show that for each smooth function on $\mathbb{R}^n$ we can construct an embedding of $\mathbb{R}^n$ to $\mathbb{L}^{n+2}$ through $\mathcal{L}$.

Example 3.30. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth real function. We define

$\phi_f\colon \mathbb{R}^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ given by

$$\phi_f(p) = (f(p), p, f(p)).$$

For every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we have

$$d(\phi_f)_p(\mathbf{v}) = (\mathbf{v}(f), \mathbf{v}, \mathbf{v}(f))$$

and

$$\langle d(\phi_f)_p(\mathbf{v}), d(\phi_f)_p(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle_{\mathbb{R}^n}.$$

That is, $\phi_f^*(\langle,\rangle) = \langle,\rangle_{\mathbb{R}^n}$ and ϕ_f determines a spacelike isometric immersion of the Euclidean space through $\mathcal{L}$. Moreover, the immersion is marginally trapped except at points where $\Delta_{\mathbb{R}^n} f = 0$ on $\mathbb{R}^n$.

At this point, from Proposition 3.29 we know that every codimension two complete spacelike submanifold factorizing through $\mathcal{L}$ is, up to an isometry, as in Example 3.30.

Corollary 3.31. Let $\psi : \Sigma^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ be a codimension two complete spacelike submanifold which factorizes through $\mathcal{L}$. Then there exists an isometry $\Psi : (\Sigma^n, \langle,\rangle) \to (\mathbb{R}^n, \langle,\rangle_{\mathbb{R}^n})$ such that $\psi = \phi_f \circ \Psi$, where $f = u \circ \Psi^{-1}$ with $u = -\langle \psi, \mathbf{e}_1\rangle = \psi_1$ and $\phi_f : \mathbb{R}^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ is the embedding

$$\phi_f(p) = (f(p), p, f(p)).$$

$$\begin{tikzcd} \Sigma^n \arrow[r, "u"] \arrow[d, shift left, "\Psi"] & \mathbb{R} \\ \mathbb{R}^n \arrow[u, shift left, "\Psi^{-1}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \Sigma^n \arrow[r, "\psi"] \arrow[d, shift left, "\Psi"] & \mathcal{L} \subset \mathbb{L}^{n+2} \\ \mathbb{R}^n \arrow[u, shift left, "\Psi^{-1}"] \arrow[ur, "\phi_f"'] & \end{tikzcd}$$

In particular, the immersion ψ is an embedding and it is marginally trapped except at points where $\Delta u = 0$ on Σ.

As a consequence, we can characterize codimension two spacelike submanifolds which factorize through $\mathcal{L}$ and that have parallel mean curvature vector as follows.

Corollary 3.32. Let $\psi : \Sigma^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ be a codimension two complete spacelike submanifold which factorizes through $\mathcal{L}$ and that has parallel mean curvature vector. Then there exists an isometry $\Psi : (\Sigma^n, \langle,\rangle) \to (\mathbb{R}^n, \langle,\rangle_{\mathbb{R}^n})$ such that $\psi = \phi_{\varrho,c} \circ \Psi$, where $\phi_{\varrho,c} : \mathbb{R}^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ is the embedding

$$\phi_{\varrho,c}(p) = (\varrho(p) + c\|p\|^2, p, \varrho(p) + c\|p\|^2)$$

for some harmonic function ϱ on $\mathbb{R}^n$ and $c \in \mathbb{R}$. Moreover:

(i) Σ is minimal if, and only if, $c = 0$.

(ii) Σ is future marginally trapped if, and only if, $c < 0$.

(iii) Σ is past marginally trapped if, and only if, $c > 0$.

Proof. Since $\langle \mathbf{a}, \mathbf{e}_1 \rangle = -1$, it follows from (3.53) that

$$\mathbf{H} = -\frac{\Delta u}{n}\mathbf{a}. \tag{3.54}$$

From (3.54), $\mathbf{H}$ is parallel if, and only if, $\Delta u =$ constant on $(\Sigma, \langle , \rangle)$. Equivalently, since $u = f \circ \Psi$ with Ψ an isometry between $(\Sigma, \langle , \rangle)$ and $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$, $\mathbf{H}$ is parallel if, and only if, $\Delta_{\mathbb{R}^n} f =$ constant on $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$.

Consider the function

$$g(p) = \frac{\Delta_{\mathbb{R}^n} f}{2n} \|p\|^2$$

for every $p \in \mathbb{R}^n$. It is satisfied

$$\nabla^{\mathbb{R}^n} g = \frac{\Delta_{\mathbb{R}^n} f}{n} p \quad \text{and} \quad \Delta_{\mathbb{R}^n} g = \Delta_{\mathbb{R}^n} f.$$

Then, defining $\varrho(p) = f(p) - g(p)$ we have $\Delta_{\mathbb{R}^n} \varrho = 0$, that is, ϱ is an harmonic function on $\mathbb{R}^n$ and $f(p) = \varrho(p) + c\|p\|^2$ where $c = \frac{\Delta_{\mathbb{R}^n} f}{2n} \in \mathbb{R}$. The last assertions follow from (3.54) since $\mathbf{H} = -\frac{c}{n}\mathbf{a}$, with $\mathbf{a}$ future-pointing. $\square$

CHAPTER 4

Codimension two spacelike submanifolds through a null hypersurface of de Sitter spacetime

In this chapter we study the case when the ambient space is the well known de Sitter spacetime. As in previous chapter, we will be focus on codimension two spacelike submanifolds which factorize through a null hypersurface of the spacetime. With this aim, we have studied two null hypersurfaces of de Sitter spacetime: the light cone and the called past infinite of the steady state space. The results included here can be found in our paper [4].

4.1 Preliminaries

Let us start by defining the $(n+2)$-dimensional de Sitter spacetime. For this, we consider it immersed in the $(n+3)$-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^{n+3}$ with coordinates $(x_0, x_1, \ldots, x_{n+2})$.

Definition 4.1. The standard model of the $(n+2)$-dimensional de Sitter spacetime is the hyperquadric

$$\mathbb{S}_1^{n+2} = \{x \in \mathbb{L}^{n+3} : \langle x, x \rangle = 1\}$$

consisting of all the unit spacelike vectors in $\mathbb{L}^{n+3}$ and endowed with the induced metric from $\mathbb{L}^{n+3}$.

De Sitter spacetime $\mathbb{S}_1^{n+2}$ is a complete, simply connected, $(n+2)$-dimensional Lorentzian manifold with constant sectional curvature 1. Therefore, $\mathbb{S}_1^{n+2}$ can be seen, in Lorentzian geometry, as the equivalent of the Euclidean sphere.

We will take on $\mathbb{S}_1^{n+2}$ the time orientation induced by the globally defined timelike vector field $\mathbf{e}_0^* \in \mathfrak{X}(\mathbb{S}_1^{n+2})$ given by

$$\mathbf{e}_0^*(x) = \mathbf{e}_0 - \langle \mathbf{e}_0, x\rangle x = \mathbf{e}_0 + x_0 x, \ \ x \in \mathbb{S}_1^{n+2},$$

where $\mathbf{e}_0 = (1, 0, \dots, 0)$. Observe that for every $x \in \mathbb{S}_1^{n+2}$,

$$\langle \mathbf{e}_0^*(x), \mathbf{e}_0^*(x)\rangle = -1 - \langle \mathbf{e}_0, x\rangle^2 \leq -1 < 0.$$

As we have said, we are interested in the case where the submanifold $\psi\colon \Sigma \to \mathbb{S}_1^{n+2} \subset \mathbb{L}^{n+3}$ is contained in certain null hypersurfaces of de Sitter spacetime. Therefore, our first aim is to obtain the globally defined future-pointing normal null frame $\{\xi, \eta\}$ on Σ following the ideas in Section 2.4. All along this chapter, ∇, $\widetilde{\nabla}$ and $\overline{\nabla}$ stand for the Levi-Civita connections of Σ, $\mathbb{S}_1^{n+2}$ and $\mathbb{L}^{n+3}$ respectively.

Let $\mathbf{e}_0^\perp$ denote the normal component of $\mathbf{e}_0$ along the submanifold, that is, for every $p \in \Sigma$ we have the following orthogonal decomposition

$$\mathbf{e}_0 = \mathbf{e}_0^\top + \mathbf{e}_0^\perp + \langle \mathbf{e}_0, \psi\rangle\psi, \tag{4.1}$$

where $\mathbf{e}_0^\top \in \mathfrak{X}(\Sigma)$ is tangent to Σ and $\mathbf{e}_0^\perp \in \mathfrak{X}^\perp(\Sigma)$ is normal to Σ. In particular,

$$\langle \mathbf{e}_0^\perp, \mathbf{e}_0^\perp\rangle = -1 - \|\mathbf{e}_0^\top\|^2 - \langle \mathbf{e}_0, \psi\rangle^2 \leq -1 < 0$$

and the vector field ν given in (2.15) has the expression

$$\nu = \frac{\mathbf{e}_0^\perp}{\|\mathbf{e}_0^\perp\|} = \frac{\mathbf{e}_0^\perp}{\sqrt{1 + \|\mathbf{e}_0^\top\|^2 + \langle \mathbf{e}_0, \psi\rangle^2}}. \tag{4.2}$$

Therefore, $\langle \xi, \nu\rangle < 0$ and using formula (2.16), the vector field

$$\eta = -\frac{\langle \mathbf{e}_0^\perp, \mathbf{e}_0^\perp\rangle}{2\langle \xi, \mathbf{e}_0\rangle^2}\xi - \frac{1}{\langle \xi, \mathbf{e}_0\rangle}\mathbf{e}_0^\perp \tag{4.3}$$

provides a second globally defined normal null vector field along the submanifold which is future-pointing and satisfies $\langle \xi, \eta\rangle = -1$.

On the other hand, it follows from the Gauss equation of the submanifold that the Riemann curvature tensor R of Σ is given by

$$R(X,Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X + A_{\mathrm{II}(X,Z)}Y - A_{\mathrm{II}(Y,Z)}X$$

for any $X, Y, Z \in \mathfrak{X}(\Sigma)$, where recall that in our convention

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

In particular, taking into account formulas (2.26) and (2.27), the Ricci tensor and the scalar curvature of Σ are given by

$$\mathrm{Ric}(X,Y) = (n-1)\langle X, Y\rangle + n\langle \mathbf{H}, \mathrm{II}(X,Y)\rangle + 2\langle (A_\xi \circ A_\eta)X, Y\rangle \tag{4.4}$$

and

$$\mathrm{Scal} = n(n-1) + n^2\langle \mathbf{H}, \mathbf{H}\rangle + 2\mathrm{tr}(A_\xi \circ A_\eta). \tag{4.5}$$

4.2 Codimension two submanifolds through the light cone

To obtain the first null hypersurface of de Sitter spacetime that we are going to study, let us cosider the light cone of the $(n+3)$-dimensional Lorentz-Minkowski spacetime with vertex at $\mathbf{a} \in \mathbb{L}^{n+3}$

$$\Lambda_{\mathbf{a}}^{\mathbb{L}^{n+3}} = \{x \in \mathbb{L}^{n+3} : \langle x - \mathbf{a}, x - \mathbf{a}\rangle = 0, x \neq \mathbf{a}\}.$$

Our purpose here is to intersect one of these subsets with de Sitter spacetime of dimension $(n+2)$. Then, take $\mathbf{a} \in \mathbb{S}_1^{n+2}$ and observe that the condition $\langle x - \mathbf{a}, x - \mathbf{a}\rangle = 0$ is equivalent to

$$\langle x, x\rangle + \langle \mathbf{a}, \mathbf{a}\rangle - 2\langle \mathbf{a}, x\rangle = 0,$$

that is, if $x \in \mathbb{S}_1^{n+2}$ and $x \in \Lambda_{\mathbf{a}}^{\mathbb{L}^{n+3}}$, then $\langle \mathbf{a}, x\rangle = 1$. We are now able to define the first null hypersurface of de Sitter spacetime where our codimension two spacelike submanifold Σ will be contained. It is called, analogously to the case of the Lorentz-Minkowski spacetime, the light cone of $\mathbb{S}_1^{n+2}$.

Definition 4.2. Let $\mathbf{a} \in \mathbb{S}_1^{n+2}$ be a fixed point of de Sitter spacetime. The light cone of $\mathbb{S}_1^{n+2}$ with vertex at $\mathbf{a}$ is the subset

$$\Lambda_{\mathbf{a}} = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x\rangle = 1, x \neq \mathbf{a}\}.$$

Figure 4.1: Light cone of de Sitter spacetime

Remark 4.3. Observe that a given point $x \in \mathbb{S}_1^{n+2}$ belongs to $\Lambda_{\mathbf{a}}$ if, and only if, $x - \mathbf{a}$ is null. That is,

$$\Lambda_{\mathbf{a}} = \{x \in \mathbb{S}_1^{n+2} : \langle x - \mathbf{a}, x - \mathbf{a}\rangle = 0, x \neq \mathbf{a}\}.$$

This subset corresponds to the subset of all the points of de Sitter spacetime which can be reached from $\mathbf{a}$ through a null (or lightlike) geodesic starting at $\mathbf{a}$ and, similarly as when we work in the light cone of the Lorentz Minkowski spacetime, we have that $\Lambda_{\mathbf{a}}$ has two connected components. We will assume that our (always connected) submanifold factorizes through the future one.

Definition 4.4. The future component of $\Lambda_{\mathbf{a}}$ consists of all points $x \in \Lambda_{\mathbf{a}}$ for which the null vector $x - \mathbf{a}$ is future-pointing, that is,

$$\Lambda_{\mathbf{a}}^{+} = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x\rangle = 1, \langle x - \mathbf{a}, \mathbf{e}_0\rangle = -x_0 + a_0 < 0\}.$$

Respectively, the past component of $\Lambda_{\mathbf{a}}$ is the subset

$$\Lambda_{\mathbf{a}}^{-} = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x\rangle = 1, \langle x - \mathbf{a}, \mathbf{e}_0\rangle = -x_0 + a_0 > 0\},$$

and it corresponds to all points $x \in \Lambda_{\mathbf{a}}$ for which the null vector $x - \mathbf{a}$ is past-pointing.

Let $\psi : \Sigma^n \to \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold and assume that $\psi(\Sigma)$ factorizes through the future connected component of the light cone with vertex at $\mathbf{a}$ for some point $\mathbf{a} \in \mathbb{S}_1^{n+2}$, that is, $\psi(\Sigma) \subset \Lambda_{\mathbf{a}}^+$.

In other words, $\psi(p) \neq \mathbf{a}$ for every $p \in \Sigma$, and it is satisfied

$$\langle \psi, \psi \rangle = 1, \quad \langle \psi, \mathbf{a} \rangle = 1 \quad \text{and} \quad \langle \psi - \mathbf{a}, \mathbf{e}_0 \rangle < 0.$$

In this case

$$\xi = \psi - \mathbf{a}$$

is a globally defined null vector field which is normal to the submanifold, tangent to de Sitter spacetime and future-pointing. Therefore, it can be chosen as the first vector field of our globally defined future-pointing normal null frame.

We define the function $u : \Sigma \to (0, +\infty)$ setting

$$u = -\langle \psi - \mathbf{a}, \mathbf{e}_0 \rangle = \psi_0 - a_0 > 0.$$

It follows that

$$\nabla u = -\mathbf{e}_0^\top$$

where we are denoting $\mathbf{e}_0 = \mathbf{e}_0^\top + \mathbf{e}_0^\perp + \langle \psi, \mathbf{e}_0 \rangle \psi$ as in (4.1).

Thus, we get the expression

$$\mathbf{e}_0 = \mathbf{e}_0^\perp - \nabla u - (u + a_0)\psi. \tag{4.6}$$

From this and (4.2) we deduce

$$\nu = \frac{1}{\sqrt{1 + \|\nabla u\|^2 + (u + a_0)^2}} (\mathbf{e}_0 + \nabla u + (u + a_0)\psi)$$

and

$$\langle \xi, \nu \rangle = \frac{\langle \psi - \mathbf{a}, \mathbf{e}_0 \rangle}{\sqrt{1 + \|\nabla u\|^2 + (u + a_0)^2}} = -\frac{u}{\sqrt{1 + \|\nabla u\|^2 + (u + a_0)^2}} < 0.$$

Therefore, from (4.3) we obtain

$$\begin{aligned} \eta &= -\frac{1 + \|\nabla u\|^2 + (u + a_0)^2}{2u^2} (\psi - \mathbf{a}) + \frac{1}{u} (\mathbf{e}_0 + \nabla u + (u + a_0)\psi) \\ &= -\frac{1 + \|\nabla u\|^2 + (u + a_0)^2}{2u^2} \xi + \frac{1}{u} \mathbf{e}_0^\perp. \end{aligned} \tag{4.7}$$

We summarize the discussion above in the following result.

Proposition 4.5. Let $\psi : \Sigma^n \to \Lambda_{\mathbf{a}}^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of the light cone $\Lambda_{\mathbf{a}}^+$. Then,

$$\xi = \psi - \mathbf{a} \qquad \text{and} \qquad \eta = -\frac{1 + \|\nabla u\|^2 + (u + a_0)^2}{2u^2}\xi + \frac{1}{u}\mathbf{e}_0^\perp \tag{4.8}$$

are two globally defined normal null vector fields along the submanifold which are future-pointing and satisfy $\langle \xi, \eta \rangle = -1$.

Next we compute the associated shape operators.

Proposition 4.6. Let $\psi : \Sigma^n \to \Lambda_{\mathbf{a}}^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of the light cone $\Lambda_{\mathbf{a}}^+$. Then,

$$A_\xi = I \qquad \text{and} \qquad A_\eta = -\frac{1 + \|\nabla u\|^2 - u^2 + a_0^2}{2u^2} I + \frac{1}{u}\nabla^2 u \tag{4.9}$$

are the shape operators associated to ξ and η respectively. In particular, the null mean curvatures are given by

$$\theta_\xi = 1 \quad \text{and} \quad \theta_\eta = \frac{2u\Delta u - n(1 + \|\nabla u\|^2 - u^2 + a_0^2)}{2nu^2} \tag{4.10}$$

where recall that ∇^2 and Δ stands respectively for the Hessian and the Laplacian operators defined in Section 2.1.

Proof. Taking into account Weingarten formula (2.5), we get

$$\overline{\nabla}_X \xi = X = A_\xi X + \nabla_X^\perp \xi$$

for every $X \in \mathfrak{X}(\Sigma)$, so that $A_\xi X = X$ and $\nabla_X^\perp \xi = 0$. To obtain the expression for A_η we observe from (4.7) that

$$A_\eta = -\frac{1 + \|\nabla u\|^2 + (u + a_0)^2}{2u^2} A_\xi + \frac{1}{u} A_{\mathbf{e}_0^\perp}.$$

Using formula (4.1), for every $X \in \mathfrak{X}(\Sigma)$

$$0 = \widetilde{\nabla}_X \mathbf{e}_0 = \widetilde{\nabla}_X \mathbf{e}_0^\perp - \widetilde{\nabla}_X \nabla u - \widetilde{\nabla}_X((u + a_0)\psi). \tag{4.11}$$

On the other hand, we compute

$$\widetilde{\nabla}_X \mathbf{e}_0^\perp = \nabla_X^\perp \mathbf{e}_0^\perp + A_{\mathbf{e}_0^\perp} X,$$

$$\widetilde{\nabla}_X \nabla u = \overline{\nabla}_X \nabla u - \langle X, \nabla u \rangle \psi = \nabla_X \nabla u - \mathrm{II}(\nabla u, X) - X(u)\psi,$$

and

$$\widetilde{\nabla}_X ((u + a_0)\psi) = X(u)\psi + (u + a_0)X,$$

where we have used Gauss and Weingarten formulas (2.4) and (2.5). Inserting this into (4.11) we obtain

$$0 = \nabla_X^\perp \mathbf{e}_0^\perp + A_{\mathbf{e}_0^\perp} X - \nabla_X \nabla u + \mathrm{II}(\nabla u, X) - (u + a_0)X,$$

and, in particular,

$$A_{\mathbf{e}_0^\perp} X = \nabla_X \nabla u + (u + a_0)X.$$

Therefore, we have

$$\begin{aligned} A_\eta &= -\frac{1 + \|\nabla u\|^2 + (u + a_0)^2}{2u^2} I + \frac{1}{u}(\nabla^2 u + (u + a_0)I) \\ &= -\frac{1 + \|\nabla u\|^2 - u^2 + a_0^2}{2u^2} I + \frac{1}{u} \nabla^2 u \end{aligned}$$

as we wanted to prove. Finally, taking traces in the expressions for A_ξ and A_η we obtain (4.10). □

Remark 4.7. Observe that, in the same way that happens with the global null frame obtained in the Lorentz-Minkowski spacetime, the global null frame $\{\xi, \eta\}$ given in Proposition 4.5 is parallel in the normal bundle and, in particular, the normal connection is flat.

Using Proposition 4.6 and formula (2.24), the mean curvature vector field of a codimension two spacelike submanifold Σ which factorizes through the future component of a light cone $\Lambda_{\mathbf{a}}^+$ of de Sitter spacetime is given by

$$\mathbf{H} = -\frac{1}{2nu^2}(2u\Delta u - n(1 + \|\nabla u\|^2 - u^2 + a_0^2))\xi - \eta,$$

and, in particular,

$$\langle \mathbf{H}, \mathbf{H} \rangle = -\frac{1}{nu^2} \left(2u\Delta u - n(1 + \|\nabla u\|^2 - u^2 + a_0^2)\right). \tag{4.12}$$

Therefore, Σ is marginally trapped if, and only if,

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2 + a_0^2) = 0$$

on Σ, and in that case Σ is necessarily past marginally trapped since $\theta_\xi = 1 > 0$ (see Subsection 2.3.1). By formulas (4.4) and (4.5) we directly obtain the expressions for the Ricci tensor and the scalar curvature of Σ,

$$\begin{aligned}\mathrm{Ric}(X,Y) &= (n-1)(1 + \langle \mathbf{H}, \mathbf{H} \rangle)\langle X, Y \rangle \\ &\quad + \frac{n-2}{nu}(\Delta u \langle X, Y \rangle - n \mathrm{Hess}\, u(X,Y)),\end{aligned} \tag{4.13}$$

and

$$\mathrm{Scal} = n(n-1)(1 + \langle \mathbf{H}, \mathbf{H} \rangle). \tag{4.14}$$

As a consequence of these computations we have next corollary.

Corollary 4.8. Let $\psi : \Sigma^n \to \Lambda_{\mathbf{a}}^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through the future component of a light cone $\Lambda_{\mathbf{a}}^+$. The following assertions are equivalent:

(i) Σ is (necessarily past) marginally trapped.

(ii) The positive function $u = -\langle \psi - \mathbf{a}, \mathbf{e}_0 \rangle$ satisfies the differential equation

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2 + a_0^2) = 0 \quad \text{on} \quad \Sigma. \tag{4.15}$$

(iii) Σ has constant scalar curvature $\mathrm{Scal} = n(n-1)$.

Remark 4.9. Notice that we also have that Σ is trapped if, and only if, $2u\Delta u - n(1 + \|\nabla u\|^2 - u^2 + a_0^2) > 0$, and, therefore, Σ is weakly trapped if, and only if, $2u\Delta u - n(1 + \|\nabla u\|^2 - u^2 + a_0^2) \geq 0$.

4.2.1 Examples

This subsection is devoted to show some examples of codimension two spacelike submanifolds which factorize through $\Lambda_{\mathbf{a}}^+$. They have been obtained from [15].

Example 4.10. Let us take $\psi : \mathbb{R}^2 \to \mathbb{S}_1^4 \subset \mathbb{L}^5$ the map given by

$$\psi(x,y) = (1, \sin(x), \cos(x)\cos(y), \cos(x)\sin(y), 1).$$

Notice that ψ factorizes through the light cone with vertex at $\mathbf{a} = (0, \dots, 0, 1)$. Consider now $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ and compute,

$$d\psi_{(x,y)}(\mathbf{v}) = (0, v_1 \cos(x), -v_1 \sin(x)\cos(y) - v_2 \cos(x)\sin(y),\\ -v_1 \sin(x)\sin(y) + v_2 \cos(x)\cos(y), 0).$$

Then, if $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$, it is satisfied

$$\langle d\psi_{(x,y)}(\mathbf{v}), d\psi_{(x,y)}(\mathbf{w})\rangle = v_1 w_1 + \cos^2(x) v_2 w_2,$$

that is,

$$\psi^*(\langle , \rangle) = dx^2 + \cos^2(x) dy^2.$$

It means that the induced metric is a warped metric with warping function $\cos(x)$. Defining $u(x,y) = -\langle \psi(x,y) - \mathbf{a}, \mathbf{e_0}\rangle = 1$, for $(x,y) \in \mathbb{R}^2$, we can easily check that $\nabla u = 0$ and $\Delta u = 0$. Consequently, by assertion (ii) in Corollary 4.8, the submanifold is marginally trapped.

Next example shows a codimension two spacelike submanifold through the light cone which is trapped.

Example 4.11. Define $\psi : \mathbb{R}^2 \to \mathbb{S}^4_1 \subset \mathbb{L}^5$ the map given by

$$\psi(x,y) = (\cosh(x)\cosh(y), \cosh(x)\sinh(y), \sinh(x), 1, 1),$$

which, as in the previous example, factorizes through the light cone with vertex at $\mathbf{a} = (0, \dots, 0, 1)$. Take $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ and compute

$$d\psi_{(x,y)}(\mathbf{v}) = (v_1 \sinh(x)\cosh(y) + v_2 \cosh(x)\sinh(y),\\ v_1 \sinh(x)\sinh(y) + v_2 \cosh(x)\cosh(y), v_1 \cosh(x), 0, 0).$$

From here, letting $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ we obtain

$$\langle d\psi_{(x,y)}(\mathbf{v}), d\psi_{(x,y)}(\mathbf{w})\rangle = v_1 w_1 + \cosh^2(x) v_2 w_2,$$

that is, the induced metric by ψ is the warped metric

$$\psi^*(\langle , \rangle) = dx^2 + \cosh^2(x) dy^2. \tag{4.16}$$

We define now the positive function

$$u(x,y) = -\langle \psi(x,y) - \mathbf{a}, \mathbf{e}_0\rangle = \cosh(x)\cosh(y)$$

and, in order to check if it satisfies the differential equation (4.15), we compute

∇u and Δu with respect the metric obtained in (4.16). For this, observe that the the matrix of the metric is written as

$$[g_{i,j}] = \begin{bmatrix} 1 & 0 \\ 0 & \cosh^2(x) \end{bmatrix}$$

and its inverse is

$$\left[g^{i,j}\right] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\cosh^2(x)} \end{bmatrix}.$$

On the other hand, we compute

$$\frac{\partial u}{\partial x} = \sinh(x)\cosh(y) \quad \text{and} \quad \frac{\partial u}{\partial y} = \cosh(x)\sinh(y).$$

From here, and using formula (2.1), we get

$$\nabla u = (\sinh(x)\cosh(y), \frac{\sinh(y)}{\cosh(y)})$$

and

$$\begin{aligned} \|\nabla u\|^2 &= \sinh^2(x)\cosh^2(y) + \sinh^2(y) \\ &= -1 + \cosh^2(x)\cosh^2(y) = -1 + u^2. \end{aligned} \tag{4.17}$$

In order to obtain the expression of Δu, and taking into account that $\det(g_{i,j}) = \cosh^2(x)$, we compute

$$\frac{\partial}{\partial x}(g^{1,1}\frac{\partial u}{\partial x}) = \cosh(y)(\sinh^2(x) + \cosh^2(x))$$

and

$$\frac{\partial}{\partial y}(g^{2,2}\frac{\partial u}{\partial y}) = \cosh(y).$$

Then, by formula (2.3), it follows

$$\begin{aligned} \Delta u &= \frac{\cosh(y)}{\cosh(x)}(\sinh^2(x) + \cosh^2(x) + 1) \\ &= 2\cosh(x)\cosh(y) = 2u, \end{aligned}$$

and, finally, we conclude

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 2u\Delta u = 4u^2 > 0.$$

Then, taking into account Remark 4.9, the submanifold is trapped.

4.3 Characterization of compact marginally trapped submanifolds through the light cone

In what follows, and without loss of generality, we may assume that the vertex of the light cone is the point $\mathbf{a} = (0, \dots, 0, 1) \in \mathbb{S}_1^{n+2}$, so that

$$\Lambda^+ = \{x \in \mathbb{S}_1^{n+2} : x_{n+2} = 1, x_0 > 0\}.$$

As all along this chapter, let us denote by u the positive function on Σ given by $u = -\langle \psi - \mathbf{a}, \mathbf{e}_0 \rangle = -\langle \psi, \mathbf{e}_0 \rangle = \psi_0 > 0$. Before showing our main results, and since we will use it again, we recall here Lemma 3.11.

Lemma 4.12. Let $\langle , \rangle$ be a complete metric on a Riemannian manifold Σ and let r denote the Riemannian distance function from a fixed origin $o \in \Sigma$. If a function w satisfies

$$w^{2/(n-2)}(p) \geq \frac{C}{r(p)\log(r(p))}, \quad r(p) \gg 1, \tag{4.18}$$

with C a positive constant, then the conformal metric $\widetilde{\langle , \rangle} = w^{4/(n-2)} \langle , \rangle$ is also complete.

Now we are ready to prove our first result, which is the equivalent to Proposition 3.12 in the Lorentz-Minkowski spacetime.

Proposition 4.13. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through Λ^+. Assume that Σ is complete and that the positive function u satisfies

$$u(p) \leq C\, r(p) \log(r(p)), \quad r(p) \gg 1 \tag{4.19}$$

where C is a positive constant and r denotes the Riemannian distance function from a fixed origin $o \in \Sigma$. Then Σ is compact and conformally diffeomorphic to the round sphere $\mathbb{S}^n$. In particular, this holds if $\sup_\Sigma u < +\infty$ and, more generally, if $\limsup\limits_{r \to +\infty} \dfrac{u}{r \log(r)} < +\infty$.

Proof. Observe that, for every $p \in \Sigma$, $\psi(p) = (u(p), \psi_1(p), \dots, \psi_{n+1}(p), 1)$, where

$$\sum_{i=1}^{n+1} \psi_i^2(p) = u^2(p) > 0. \tag{4.20}$$

We define the function $\Psi : \Sigma^n \to \mathbb{S}^n$ by

$$\Psi(p) = \frac{1}{u(p)}(\psi_1(p), \ldots, \psi_{n+1}(p)),$$

and then, for every $p \in \Sigma$ and $\mathbf{v} \in T_p\Sigma$ we obtain

$$d\Psi_p(\mathbf{v}) = -\frac{\mathbf{v}(u)}{u^2(p)}(\psi_1(p), \ldots, \psi_{n+1}(p)) + \frac{1}{u(p)}(\mathbf{v}(\psi_1), \ldots, \mathbf{v}(\psi_{n+1})).$$

Denote by $\langle , \rangle_0$ the standard metric of the round sphere $\mathbb{S}^n$. Therefore, for every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we have

$$\begin{aligned}
\langle d\Psi_p(\mathbf{v}), d\Psi_p(\mathbf{w})\rangle_0 &= \frac{\mathbf{v}(u)\mathbf{w}(u)}{u^4(p)} \sum_{i=1}^{n+1} \psi_i^2(p) + \frac{1}{u^2(p)} \sum_{i=1}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) \\
&\quad - \frac{\mathbf{v}(u)}{2u^3(p)}\mathbf{w}\left(\sum_{i=1}^{n+1} \psi_i^2\right) - \frac{\mathbf{w}(u)}{2u^3(p)}\mathbf{v}\left(\sum_{i=1}^{n+1} \psi_i^2\right) \\
&= \frac{1}{u^2(p)}\left(-\mathbf{v}(u)\mathbf{w}(u) + \sum_{i=1}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i)\right) \\
&= \frac{1}{u^2(p)}\langle d\psi_p(\mathbf{v}), d\psi_p(\mathbf{w})\rangle = \frac{1}{u^2(p)}\langle \mathbf{v}, \mathbf{w}\rangle.
\end{aligned}$$

In other words,

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle, \tag{4.21}$$

where recall that by $\langle , \rangle$ we denote the Riemannian metric on Σ induced by the immersion ψ.

From (4.21) it follows that Ψ is a local diffeomorphism. Assume now that Σ is complete (that is, $\langle , \rangle$ is a complete Riemannian metric on Σ) and u satisfies (4.19). Therefore, by Lemma 4.12 applied to the function $w = u^{-(n-2)/2}$, we know that the conformal metric

$$\widetilde{\langle , \rangle} = \frac{1}{u^2}\langle , \rangle$$

is also complete on Σ. Then, equation (4.21) means that the map

$$\Psi : (\Sigma^n, \widetilde{\langle,\rangle}) \to (\mathbb{S}^n, \langle,\rangle_0)$$

is a local isometry between complete Riemannian manifolds, and following the same reasoning that in Proposition 3.12, Ψ is a global diffeomorphism and this ends the proof. $\square$

Example 4.14. In this example we observe that for each positive smooth function $f : \mathbb{S}^n \to (0, +\infty)$ we can construct an embedding $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ by setting

$$\psi_f(p) = (f(p), f(p)p, 1).$$

Clearly, for every $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^n$ we have

$$d(\psi_f)_p(\mathbf{v}) = (\mathbf{v}(f), \mathbf{v}(f)p + f(p)\mathbf{v}, 0)$$

and

$$\langle d(\psi_f)_p(\mathbf{v}), d(\psi_f)_p(\mathbf{w})\rangle = f^2(p)\langle \mathbf{v}, \mathbf{w}\rangle_0.$$

That is,

$$\langle,\rangle = \psi_f^*(\langle,\rangle) = f^2\langle,\rangle_0, \tag{4.22}$$

which means that ψ_f determines a spacelike immersion of $\mathbb{S}^n$ through Λ^+ whose induced metric is conformal to the standard metric of the round sphere.

In this case $u = f$ and, similarly as we did in Example 3.14 in the Lorentz-Minkowski spacetime, we can obtain the expression of the second fundamental form of ψ_f in terms of the function f and the gradient and the Hessian of f with respect to the round metric $\langle,\rangle_0$. First of all recall that $A_\xi = I$ and $\theta_\xi = 1$. Now, to compute A_η, we use (2.11) and (2.12), and we conclude

$$A_\eta(X) = \frac{1}{f^3}\nabla^0_X\nabla^0 f - \frac{2}{f^4}\langle X, \nabla^0 f\rangle_0 \nabla^0 f + \frac{\|\nabla^0 f\|_0^2 + f^4 - f^2}{2f^4}X \tag{4.23}$$

for every $X \in \mathfrak{X}(\mathbb{S}^n)$. Thus, tracing (4.23) with respect to $\langle,\rangle_0$ we have

$$\theta_\eta = \frac{2f\Delta_0 f + (n-4)\|\nabla^0 f\|_0^2 - nf^2(1-f^2)}{2nf^4}, \tag{4.24}$$

and, in particular, ψ_f is marginally trapped if, and only if,

$$2f\Delta_0 f + (n-4)\|\nabla^0 f\|_0^2 - nf^2(1-f^2) = 0 \quad \text{on } (\mathbb{S}^n, \langle,\rangle_0). \tag{4.25}$$

The following result is a direct consequence of Proposition 4.13. Here we state that every codimension two compact spacelike submanifold factorizing through Λ^+ is, up to a conformal diffeomorphism, as in Example 4.10.

Corollary 4.15. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two compact spacelike submanifold which factorizes through Λ^+. Then there exists a conformal diffeomorphism $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ such that

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle$$

with $u = -\langle \psi, \mathbf{e}_0 \rangle = \psi_0 > 0$, and $\psi = \psi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$ and $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ is the embedding

$$\psi_f(p) = (f(p), f(p)p, 1).$$

$$\begin{tikzcd} \Sigma^n \arrow[r, "u"] \arrow[d, shift left, "\Psi"] & (0,+\infty) \\ \mathbb{S}^n \arrow[u, shift left, "\Psi^{-1}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \Sigma^n \arrow[r, "\psi"] \arrow[d, shift left, "\Psi"] & \Lambda^+ \subset \mathbb{S}_1^{n+2} \\ \mathbb{S}^n \arrow[u, shift left, "\Psi^{-1}"] \arrow[ur, "\psi_f"'] & \end{tikzcd}$$

In particular, the immersion ψ is an embedding.

To see that, simply consider u and Ψ as in the proof of Proposition 4.13, and recall that in this situation $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ is a conformal diffeomorphism with

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle.$$

Let $\Phi : \mathbb{S}^n \to \Sigma^n$ be the inverse of Ψ. Then taking $f = u \circ \Phi$ one has $f \circ \Psi = u$ and $\psi = \psi_f \circ \Psi$, since

$$\begin{aligned} \psi_f \circ \Psi(p) &= (f(\Psi(p)), f(\Psi(p))\Psi(p), 1) \\ &= (u(p), \psi_1(p), \dots, \psi_{n+1}(p), 1) = \psi(p). \end{aligned}$$

Motivated by the above discussion we consider the following example.

Example 4.16. For every fixed vector $\mathbf{b} \in \mathbb{R}^{n+1}$, let $f_{\mathbf{b}} : \mathbb{S}^n \to (0, +\infty)$ be the function

$$f_{\mathbf{b}}(p) = \frac{1}{\langle p, \mathbf{b} \rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

where $\langle , \rangle_0$ stands both for the Euclidean metric in $\mathbb{R}^{n+1}$ and for the induced standard metric on the Euclidean sphere $\mathbb{S}^n$. Note that for $\mathbf{b} = \mathbf{0} \in \mathbb{R}^{n+1}$ we have $f_{\mathbf{0}} \equiv 1$. We claim that the corresponding embedding

$$\psi_{\mathbf{b}} := \psi_{f_{\mathbf{b}}} : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$$

is a (necessarily past) marginally trapped submanifold. To see it, it suffices to check the validity of (4.25) for $f = f_{\mathbf{b}}$.

Writting $f_{\mathbf{b}} = 1/g$ with

$$g(p) = \langle p, \mathbf{b} \rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}$$

we know that

$$\|\nabla^0 f_{\mathbf{b}}\|_0^2 = \frac{1}{g^4} \|\nabla^0 g\|_0^2, \quad \text{and} \quad \Delta_0 f_{\mathbf{b}} = -\frac{1}{g^2} \Delta_0 g + \frac{2}{g^3} \|\nabla^0 g\|_0^2. \tag{4.26}$$

Observe that

$$\nabla^0 g(p) = \mathbf{b} - \langle p, \mathbf{b} \rangle_0 p \quad \text{and} \quad \|\nabla^0 g\|_0^2 = \|\mathbf{b}\|_0^2 - \langle p, \mathbf{b} \rangle_0^2, \tag{4.27}$$

and then,

$$\|\nabla f_{\mathbf{b}}(p)\|_0^2 = \frac{\|\mathbf{b}\|_0^2 - \langle p, \mathbf{b} \rangle_0^2}{g^4(p)} \tag{4.28}$$

for every $p \in \mathbb{S}^n$. Furthermore,

$$\nabla_{\mathbf{v}}^0 \nabla^0 g = -\langle p, \mathbf{b} \rangle_0 \mathbf{v}$$

for every $\mathbf{v} \in T_p\mathbb{S}^n$, so that $\Delta_0 g(p) = -n\langle p, \mathbf{b} \rangle_0$. Substituting this into (4.26) and using (4.27) we obtain

$$\Delta_0 f_{\mathbf{b}}(p) = \frac{n\langle p, \mathbf{b} \rangle_0}{g^2(p)} + \frac{2\left(\|\mathbf{b}\|_0^2 - \langle p, \mathbf{b} \rangle_0^2\right)}{g^3(p)},$$

so that

$$2f_{\mathbf{b}}(p)\Delta_0 f_{\mathbf{b}}(p) = \frac{2n\langle p, \mathbf{b} \rangle_0}{g^3(p)} + \frac{4\left(\|\mathbf{b}\|_0^2 - \langle p, \mathbf{b} \rangle_0^2\right)}{g^4(p)}. \tag{4.29}$$

Finally, from (4.28) and (4.29) we get the validity of (4.25) for $f = f_{\mathbf{b}}$.

We now come to our main classification result, which shows that the above examples are in fact the only examples of codimension two compact marginally trapped submanifolds which factorize through $\Lambda^+ \subset \mathbb{S}_1^{n+2}$.

Theorem 4.17. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two compact (necessarily past) marginally trapped spacelike submanifold factorizing through Λ^+. Then there exists a conformal diffeomorphism $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ such that $\psi = \psi_{\mathbf{b}} \circ \Psi$, where $f_{\mathbf{b}} : \mathbb{S}^n \to (0, +\infty)$ is

$$f_{\mathbf{b}}(p) = \frac{1}{\langle p, \mathbf{b} \rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

for any $\mathbf{b} \in \mathbb{R}^{n+1}$ and $\psi_{\mathbf{b}} : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ is the embedding

$$\psi_{\mathbf{b}}(p) = (f_{\mathbf{b}}(p), f_{\mathbf{b}}(p)p, 1).$$

In particular, Σ is embedded.

Proof. After Corollary 4.8 and Corollary 4.15 (and following the notation therein), the proof of Theorem 4.17 reduces to solve the differential equation

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0$$

on $(\Sigma^n, \langle , \rangle)$. By our previous discussion, this is equivalent to determine the positive solutions of the differential equation

$$2f\Delta f - n(1 + \|\nabla f\|^2 - f^2) = 0 \tag{4.30}$$

on $(\mathbb{S}^n, \langle , \rangle)$, where $\langle , \rangle = f^2 \langle , \rangle_0$. Here we are denoting by $\| \cdot \|^2$, ∇ and Δ the norm, the gradient and the Laplacian operator on $\mathbb{S}^n$ with respect to the metric $\langle , \rangle$. We also know from Corollary 4.8 that $(\Sigma^n, \langle , \rangle)$ has constant scalar curvature $n(n-1)$, and hence the same is true of $(\mathbb{S}^n, \langle , \rangle)$. From a classical result by Obata [40], a conformal metric on the Euclidean sphere $\mathbb{S}^n$ has constant scalar curvature $n(n-1)$ if, and only if, it has constant sectional curvature 1. Therefore, $(\mathbb{S}^n, \langle , \rangle)$ has constant sectional curvature 1.

Summing up, and surprisingly, the problem of characterizing compact marginally trapped submanifolds through the light cone of de Sitter spacetime becomes equivalent to solving the Yamabe problem on the unit round sphere; that is, determining the positive functions f on $\mathbb{S}^n$ for which the conformal metric $f^2 \langle , \rangle_0$ has constant sectional curvature 1.

This problem was solved by Obata in [40] (see also [41]), who proved that the conformal metric $f^2 \langle , \rangle_0$ is obtained from $\langle , \rangle_0$ by a conformal diffeomorphism of the unit round sphere. In particular, the conformal factor f is the conformal factor of a conformal diffeomorphism of the unit round sphere. At this point recall (see, for instance, [38]) that, up to orthogonal transformations, every conformal diffeomorphism of $(\mathbb{S}^n, \langle , \rangle_0)$ is given by

$$F_{\mathbf{c}}(p) = \frac{p + (\mu\langle p, \mathbf{c}\rangle_0 + \lambda)\mathbf{c}}{\lambda(1 + \langle p, \mathbf{c}\rangle_0)}$$

for all $p \in \mathbb{S}^n$, where $\mathbf{c} \in \mathbb{B}^{n+1}$, denoting by $\mathbb{B}^{n+1}$ the open unit ball in $\mathbb{R}^{n+1}$,

$$\lambda = (1 - \|\mathbf{c}\|_0^2)^{-1/2} \quad \text{and} \quad \mu = (\lambda - 1)\|\mathbf{c}\|_0^2.$$

A direct computation shows that for every point $p \in \mathbb{S}^n$ and any tangent vectors $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^n$ we have

$$\langle dF_p(\mathbf{v}), dF_p(\mathbf{w})\rangle_0 = \frac{1 - \|\mathbf{c}\|_0^2}{(1 + \langle p, \mathbf{c}\rangle_0)^2}\langle \mathbf{v}, \mathbf{w}\rangle_0.$$

Therefore, the conformal factor f is given by

$$f(p) = \frac{\sqrt{1 - \|\mathbf{c}\|_0^2}}{1 + \langle p, \mathbf{c}\rangle_0} = \frac{1}{\langle p, \mathbf{b}\rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

for any $\mathbf{c} \in \mathbb{B}^{n+1}$, where

$$\mathbf{b} = \frac{\mathbf{c}}{\sqrt{1 - \|\mathbf{c}\|_0^2}} \in \mathbb{R}^{n+1},$$

completing the proof of the theorem. $\square$

Remark 4.18. Notice that Theorem 4.17 can be also proved directly without using the explicit solution of the Yamabe problem. Indeed, using the same notation that in the proof, we know that both Σ and $\mathbb{S}^n$ have constant sectional curvature 1 and constant scalar curvature $n(n-1)$. In particular, the Ricci curvature of $(\mathbb{S}^n, \langle , \rangle)$ is given by

$$\mathrm{Ric}(X, Y) = (n-1)\langle X, Y\rangle$$

and taking into account that $\langle \mathbf{H}, \mathbf{H}\rangle = 0$, it follows from (4.13) that

$$\mathrm{Hess}\, f = \frac{1}{n}\Delta f \langle , \rangle \qquad \text{on } (\mathbb{S}^n, \langle , \rangle). \tag{4.31}$$

We can rewrite this equation on $\mathbb{S}^n$ with respect to the standard metric $\langle , \rangle_0$, obtaining

$$\mathrm{Hess}_0 f = \frac{2}{f} df \otimes df + \frac{1}{n}\left(\Delta_0 f - \frac{2}{f}\|\nabla^0 f\|_0^2\right)\langle , \rangle_0. \tag{4.32}$$

Summing up, our problem reduces to find the solutions of (4.25) that satisfy (4.32). To do this, we will follow Obata's ideas in [43]. Considering the function $g = 1/f$, one has that (4.25) and (4.32) become, respectively,

$$2g\Delta_0 g - n(1 + \|\nabla^0 g\|_0^2 - g^2) = 0 \tag{4.33}$$

and

$$\mathrm{Hess}_0 g = \frac{1}{n}\Delta_0 g\langle , \rangle_0 \tag{4.34}$$

on $\mathbb{S}^n$ with respect to the standard metric $\langle , \rangle_0$. Equation (4.34) is equivalent to

$$\nabla^0_X \nabla^0 g = \frac{\Delta_0 g}{n} X$$

for every tangent vector field $X \in \mathfrak{X}(\mathbb{S}^n)$, which implies that $\nabla^0 g$ is a conformal vector field on the round sphere. It is well known (see for instance [54, Chapter 1]) that if a vector field Y is conformal on an n-dimensional Riemannian manifold, then the derivative of the scalar curvature Scal is given by

$$Y(\mathrm{Scal}) = -\frac{2(n-1)}{n}\Delta(\mathrm{div}(Y)) - \frac{2}{n}\mathrm{div}(Y)\mathrm{Scal}.$$

Applying this formula to the conformal field $\nabla^0 g$ on the round sphere $\mathbb{S}^n$ we obtain that

$$\Delta_0(\mathrm{div}_0(\nabla^0 g)) = -n\mathrm{div}_0(\nabla^0 g),$$

that is,

$$\Delta_0(\Delta_0 g + ng) = 0.$$

This implies that

$$\Delta_0 g + ng = \text{constant}.$$

In other words, the Laplacian of g satisfies

$$\Delta_0 g = -n(g - c)$$

for a certain constant $c \in \mathbb{R}$. We define now the function $\phi = g - c$. From the previous identity we obtain

$$\Delta_0 \phi + n\phi = 0,$$

what implies that either $\phi \equiv 0$ or $\phi \in \mathrm{Spec}(\mathbb{S}^n, \langle , \rangle_0)$ is a first eigenfunction of the round sphere. In the first case $g \equiv c$ is constant and, by (4.33) it must be $g \equiv 1$. In the second case, as it is well known, $\phi(p) = \langle p, \mathbf{b}\rangle_0$ for some fixed vector $\mathbf{b} \in \mathbb{R}^{n+1}$, $\mathbf{b} \neq 0$. It then follows that $g(p) = \langle p, \mathbf{b}\rangle_0 + c$ and, by (4.33) it must be $c = \sqrt{1 + \|\mathbf{b}\|_0^2}$, as desired.

Remark 4.19. It is worth pointing out that although all the embeddings $\psi_{\mathbf{b}}$ given in Example 4.16 are conformal to the round sphere and have the same constant sectional curvature 1, they are not congruent to each other. To see it, assume that $\psi_{\mathbf{b}_1}$ is congruent to $\psi_{\mathbf{b}_2}$ for $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^{n+1}$. Then there exists an isometry $A \in \mathrm{Iso}(\mathbb{S}_1^{n+2}) = O_1(n+3)$ which makes commutative the following diagram

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "\psi_{\mathbf{b}_1}"] \arrow[d, "\psi_{\mathbf{b}_2}"'] & \Lambda^+ \subset \mathbb{S}_1^{n+2} \arrow[dl, "A"] \\ \Lambda^+ \subset \mathbb{S}_1^{n+2} & \end{tikzcd}$$

Then

$$d(\psi_{\mathbf{b}_2})(\mathbf{v}) = d(A \circ \psi_{\mathbf{b}_1})_p(\mathbf{v}) = A(d(\psi_{\mathbf{b}_1})_p)(\mathbf{v}), \tag{4.35}$$

and, in particular,

$$f_{\mathbf{b}_2}^2(p)\|\mathbf{v}\|_0^2 = \|d(\psi_{\mathbf{b}_2})(\mathbf{v})\|^2 = \|d(\psi_{\mathbf{b}_1})(\mathbf{v})\|^2 = f_{\mathbf{b}_1}^2(p)\|\mathbf{v}\|_0^2$$

for every $p \in \mathbb{S}^n$ and $\mathbf{v} \in T_p\mathbb{S}^n$. It follows from here that $f_{\mathbf{b}_1} = f_{\mathbf{b}_2}$ or, equivalently, that $\mathbf{b_1} = \mathbf{b_2}$.

4.4 Marginally trapped submanifolds through the past infinite of the steady state universe

In this section we will consider another interesting case of null hypersurface of de Sitter spacetime. It is obtained by intersecting the spacetime with a null hyperplane of $\mathbb{L}^{n+3}$. Let us take a null vector $\mathbf{a} \in \mathbb{L}^{n+3}$, $\mathbf{a} \neq 0$ and consider the null hypersurface

$$L = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x\rangle = 0\}.$$

Without loss of generality we may assume that $\mathbf{a}$ is past-pointing, $\langle \mathbf{a}, \mathbf{e}_0\rangle > 0$. The open region

$$\mathcal{H}^{n+2} = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x\rangle > 0\}$$

forms the spacetime for the steady state model of the universe proposed by Bondi and Gold [12] and Hoyle [26], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times (cf. [52, Section 14.8] and [24, Section 5.2]). The steady state space is a non-complete manifold, being only half of de Sitter spacetime and having as boundary the null hypersurface L, which represents the past infinity of $\mathcal{H}^{n+2}$, usually denoted by $\mathcal{J}^-$. Finally, the null hypersurface through which the submanifold Σ will factorize is the following.

Definition 4.20. The past infinity of the steady state space $\mathcal{H}^{n+2}$ is the null hypersurface defined as

$$\mathcal{J}^- = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x \rangle = 0\},$$

where $\mathbf{a} \in \mathbb{L}^{n+3}$ is a past-pointing null vector.

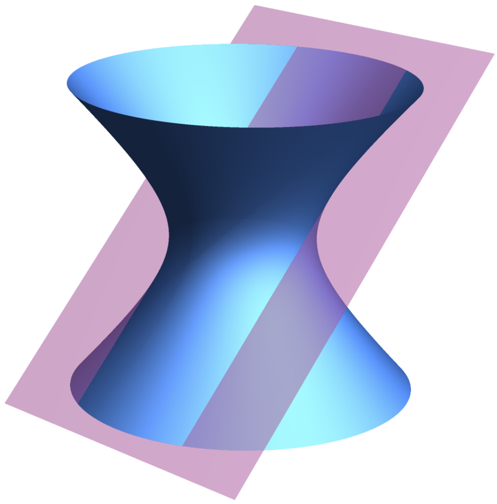

Figure 4.2: Past infinite of $\mathcal{H}^{n+2}$

Remark 4.21. Notice that Example 4.10 is not only contained in the light cone with vertex at $(0, \ldots, 0, 1)$, but it is also contained in $\mathcal{J}^-$ centered at $\mathbf{a} = (1, 0, \ldots, 0, 1)$.

Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through the past infinite $\mathcal{J}^-$ of the steady state space. In this case

$$\xi = -\mathbf{a}$$

is a future-pointing null vector field which is normal to the submanifold and hence, it can be chosen as the first vector field of our globally defined future-pointing normal null frame. We define the function $u : \Sigma \to \mathbb{R}$ by $u = -\langle \psi, \mathbf{e}_0 \rangle = \psi_0$. As in Section 4.2, $\nabla u = -\mathbf{e}_0^\top$ and ν has the expression

$$\nu = \frac{\mathbf{e}_0^\perp}{\sqrt{1 + \|\nabla u\|^2 + u^2}} = \frac{1}{\sqrt{1 + \|\nabla u\|^2 + u^2}} (\mathbf{e}_0 + \nabla u + u\psi),$$

satisfying

$$\langle \xi, \nu \rangle = -\frac{\langle \mathbf{a}, \mathbf{e}_0 \rangle}{\sqrt{1 + \|\nabla u\|^2 + u^2}} < 0.$$

Therefore, from (2.16) we obtain

$$\eta = -\frac{1 + \|\nabla u\|^2 + u^2}{2\langle \mathbf{a}, \mathbf{e}_0 \rangle^2} \xi + \frac{1}{\langle \mathbf{a}, \mathbf{e}_0 \rangle} \mathbf{e}_0^\perp,$$

and summarizing we have the following result which corresponds to Proposition 4.5 in the case of the light cone.

Proposition 4.22. Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through the past infinite $\mathcal{J}^-$ of the steady state space. Then,

$$\xi = -\mathbf{a} \qquad \text{and} \qquad \eta = -\frac{1 + \|\nabla u\|^2 + u^2}{2\langle \mathbf{a}, \mathbf{e}_0 \rangle^2} \xi + \frac{1}{\langle \mathbf{a}, \mathbf{e}_0 \rangle} \mathbf{e}_0^\perp$$

are two globally defined normal null vector fields along the submanifold which are future-pointing and satisfy $\langle \xi, \eta \rangle = -1$.

The equivalent result to Proposition 4.6 reads as follows. We omit the proof, which is similar to the one of Proposition 4.6, using now that in this setting $\overline{\nabla}_X \xi = 0$.

Proposition 4.23. Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through the past infinite $\mathcal{J}^-$ of the steady state space. Then, the shape operators associated to ξ and η are, respectively,

$$A_\xi = 0 \qquad \text{and} \qquad A_\eta = \frac{1}{\langle \mathbf{a}, \mathbf{e}_0 \rangle} \left(\nabla^2 u + uI \right).$$

In particular,

$$\theta_\xi = 0 \qquad \text{and} \qquad \theta_\eta = \frac{1}{n\langle \mathbf{a}, \mathbf{e}_0\rangle}(\Delta u + nu). \tag{4.36}$$

From the previous propositions and formula (2.24) we have that the mean curvature vector field is written as

$$\mathbf{H} = -\frac{1}{n\langle \mathbf{a}, \mathbf{e}_0\rangle}(\Delta u + nu)\xi = \frac{1}{n\langle \mathbf{a}, \mathbf{e}_0\rangle}(\Delta u + nu)\mathbf{a}, \tag{4.37}$$

and then, $\langle \mathbf{H}, \mathbf{H}\rangle = 0$. Thus, we have the following.

Proposition 4.24. Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold factorizing through $\mathcal{J}^-$. Then Σ is marginally trapped except at points where $u = -\langle \psi, \mathbf{e}_0\rangle$ satisfies $\Delta u + nu = 0$ on Σ.

In what follows, and without loss of generality, we may assume that the past-pointing null vector is $\mathbf{a} = (-1, 0, \ldots, 0, -1)$. Our next result corresponds to Proposition 4.13.

Proposition 4.25. Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two spacelike submanifold which factorizes through $\mathcal{J}^-$ and assume that Σ is complete. Then Σ is compact and isometric to the round sphere $(\mathbb{S}^n, \langle , \rangle_0)$.

Proof. Observe that, for every $p \in \Sigma$ we can write

$$\psi(p) = (u(p), \psi_1(p), \ldots, \psi_{n+1}(p), u(p))$$

where

$$\sum_{i=1}^{n+1} \psi_i^2(p) = 1.$$

Define the function $\Psi : \Sigma^n \to \mathbb{S}^n$ by

$$\Psi(p) = (\psi_1(p), \ldots \psi_{n+1}(p)).$$

For every $p \in \Sigma$ and $\mathbf{v} \in T_p\Sigma$ we obtain

$$d\Psi_p(\mathbf{v}) = (\mathbf{v}(\psi_1), \ldots, \mathbf{v}(\psi_{n+1})),$$

and therefore, for every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we have

$$\begin{aligned}
\langle d\Psi_p(\mathbf{v}), d\Psi_p(\mathbf{w})\rangle_0 &= \sum_{i=1}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) \\
&= -\mathbf{v}(u)\mathbf{w}(u) + \sum_{i=1}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) + \mathbf{v}(u)\mathbf{w}(u) \\
&= \langle d\psi_p(\mathbf{v}), d\psi_p(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle.
\end{aligned}$$

In other words, $\Psi^*(\langle , \rangle_0) = \langle , \rangle$, which means that $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ is a local isometry. Since $(\Sigma^n, \langle , \rangle$ is complete and $\mathbb{S}^n$ is simply connected, this implies that Ψ is in fact a global isometry. $\square$

The result above motivates the following example.

Example 4.26. For each smooth function $f : \mathbb{S}^n \to \mathbb{R}$ we can construct an embedding $\phi_f : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ given by

$$\phi_f(p) = (f(p), p, f(p)).$$

For every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we have

$$d(\phi_f)_p(\mathbf{v}) = (\mathbf{v}(f), \mathbf{v}, \mathbf{v}(f))$$

and

$$\langle d(\phi_f)_p(\mathbf{v}), d(\phi_f)_p(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle_0.$$

That is, $\phi_f^*(\langle , \rangle) = \langle , \rangle_0$ and ϕ_f determines a spacelike isometric immersion of the round sphere through $\mathcal{J}^-$. Moreover, the immersion is marginally trapped except at points where $\Delta_0 f + nf = 0$ on $\mathbb{S}^n$.

Actually, from Proposition 4.25 we know that every codimension two complete spacelike submanifold factorizing through $\mathcal{J}^-$ is, up to an isometry, as in previous example.

Corollary 4.27. Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two complete spacelike submanifold which factorizes through $\mathcal{J}^-$. Then Σ is compact and there exists an isometry $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ such that $\psi = \phi_f \circ \Psi$, where $f = u \circ \Psi^{-1}$ with $u = -\langle \psi, \mathbf{e}_0\rangle = \psi_0$ and $\phi_f : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ is the embedding

$$\phi_f(p) = (f(p), p, f(p)).$$

$$\begin{tikzcd}
\Sigma^n \arrow[r, "u"] \arrow[d, shift left, "\Psi"] & \mathbb{R} \\
\mathbb{S}^n \arrow[u, shift left, "\Psi^{-1}"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^n \arrow[r, "\psi"] \arrow[d, shift left, "\Psi"] & \mathcal{J}^- \subset \mathbb{S}_1^{n+2} \\
\mathbb{S}^n \arrow[u, shift left, "\Psi^{-1}"] \arrow[ur, "\phi_f"'] &
\end{tikzcd}$$

In particular, the immersion ψ is an embedding and it is marginally trapped except at points where $\Delta u + nu = 0$ on Σ.

As a consequence, we can characterize codimension two spacelike submanifolds which factorize through $\mathcal{J}^-$ and having parallel mean curvature vector as follows.

Corollary 4.28. Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two complete spacelike submanifold which factorizes through $\mathcal{J}^-$ and having parallel mean curvature vector. Then Σ is compact and there exists an isometry $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ such that $\psi = \phi_{\mathbf{b},c} \circ \Psi$, where $\phi_{\mathbf{b},c} : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ is the embedding

$$\phi_{\mathbf{b},c}(p) = (\langle p, \mathbf{b} \rangle_0 + c, p, \langle p, \mathbf{b} \rangle_0 + c).$$

for some $\mathbf{b} \in \mathbb{R}^{n+1}$ and $c \in \mathbb{R}$. Moreover:

(i) Σ is minimal if, and only if, $c = 0$.

(ii) Σ is future marginally trapped if, and only if, $c < 0$.

(iii) Σ is past marginally trapped if, and only if, $c > 0$.

Proof. Since $\langle \mathbf{a}, \mathbf{e}_0 \rangle = 1$, it follows from (4.37) that

$$\mathbf{H} = \frac{1}{n}(\Delta u + nu)\mathbf{a}. \tag{4.38}$$

Then, $\mathbf{H}$ is parallel if, and only if, $\Delta u + nu =$ constant on $(\Sigma, \langle , \rangle)$. Equivalently, since $u = f \circ \Psi$ with Ψ an isometry between $(\Sigma, \langle , \rangle)$ and $(\mathbb{S}^n, \langle , \rangle_0)$, $\mathbf{H}$ is parallel if, and only if, $\Delta_0 f + nf =$ constant on $(\mathbb{S}^n, \langle , \rangle_0)$.

Therefore, the Laplacian of f satisfies

$$\Delta_0 f = -n(f - c)$$

for a certain constant c. Defining now the function $\varrho = f - c$, the previous identity becomes

$$\Delta_0 \varrho + n\varrho = 0,$$

which implies that either $\varrho \equiv 0$ or $\varrho \in \mathrm{Spec}(\mathbb{S}^n, \langle , \rangle_0)$ is a first eigenfunction of the round sphere. In the first case $f \equiv c$ is constant (which corresponds to $\mathbf{b} = 0$). In the second case, as it is well known, $\varrho(p) = \langle p, \mathbf{b} \rangle_0$ for some fixed vector $\mathbf{b} \in \mathbb{R}^{n+1}$, $\mathbf{b} \neq 0$. It then follows that $f(p) = \langle p, \mathbf{b} \rangle_0 + c$. The last assertions follow from (4.38) since $\mathbf{H} = c\mathbf{a}$, with $\mathbf{a}$ past-pointing. $\square$

In particular, for the case of minimal submanifolds we have the following.

Corollary 4.29. Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension two complete spacelike submanifold which factorizes through $\mathcal{J}^-$. Σ is minimal if, and only if, it is compact and there exists an isometry $\Psi : (\Sigma^n, \langle,\rangle) \to (\mathbb{S}^n, \langle,\rangle_0)$ such that $\psi = \phi_{\mathbf{b}} \circ \Psi$, where $\phi_{\mathbf{b}} : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ is the embedding

$$\phi_{\mathbf{b}}(p) = (\langle p, \mathbf{b}\rangle_0, p, \langle p, \mathbf{b}\rangle_0)$$

for some $\mathbf{b} \in \mathbb{R}^{n+1}$.

4.5 A uniqueness result for the marginally trapped type equation on compact manifolds

In this section, motivated by the geometric meaning of the solutions to the partial differential equation (4.15), we have established the following intrinsic uniqueness result.

Theorem 4.30. Let $(\Sigma, \langle,\rangle)$ be a compact Riemannian manifold of dimension $n \geq 2$ and Ricci curvature satisfying

$$\mathrm{Ric} \geq K \tag{4.39}$$

for some constant $K > (n-1)$. The only positive solution to the partial differential equation

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0 \quad \text{on } \Sigma \tag{4.40}$$

is the constant function $u \equiv 1$.

Proof. Consider the vector field

$$V = u^{-(n-1)} \left(\frac{1}{2} \nabla \|\nabla u\|^2 - \frac{\Delta u}{n} \nabla u \right).$$

The divergence of V is given by

$$\begin{aligned} \mathrm{div}(V) =& u^{-(n-1)} \left(\frac{1}{2} \Delta \|\nabla u\|^2 - \frac{1}{n}((\Delta u)^2 + \langle \nabla \Delta u, \nabla u\rangle) \right) \\ &- \frac{n-1}{2} u^{-n} \langle \nabla \|\nabla u\|^2, \nabla u\rangle + \frac{n-1}{n} u^{-n} \Delta u \|\nabla u\|^2. \end{aligned} \tag{4.41}$$

Bochner-Lichnerowicz formula states that

$$\frac{1}{2}\Delta\|\nabla u\|^2 = \|\nabla^2 u\|^2 + \langle \nabla\Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u),$$

and putting this into (4.41) we have

$$\begin{aligned} \mathrm{div}(V) =& u^{-(n-1)}\left(\|\nabla^2 u\|^2 + \mathrm{Ric}(\nabla u, \nabla u) - \frac{(\Delta u)^2}{n}\right) \\ &+ \frac{n-1}{n}u^{-(n-1)}\langle \nabla\Delta u, \nabla u\rangle - \frac{n-1}{2}u^{-n}\langle \nabla\|\nabla u\|^2, \nabla u\rangle \\ &+ \frac{n-1}{n}u^{-n}\Delta u\|\nabla u\|^2. \end{aligned} \tag{4.42}$$

Using now (4.40), we have that

$$\Delta u = \frac{n}{2}u^{-1}\left(1 - u^2 + \|\nabla u\|^2\right) \tag{4.43}$$

and from here,

$$\nabla\Delta u = -\frac{n}{2}u^{-2}(1 + u^2 + \|\nabla u\|^2)\nabla u + \frac{n}{2}u^{-1}\nabla\|\nabla u\|^2. \tag{4.44}$$

Taking into account (4.43) and (4.44), the expression (4.42) becomes

$$\begin{aligned} \mathrm{div}(V) = u^{-(n-1)}&\left(\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n}\right) \\ &+ u^{-(n-1)}\left(\mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2\right). \end{aligned} \tag{4.45}$$

Then, integrating and using the divergence theorem we obtain

$$\begin{aligned} &\int_\Sigma u^{-(n-1)}\left(\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n}\right) \\ &+ \int_\Sigma u^{-(n-1)}\left(\mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2\right) = 0. \end{aligned} \tag{4.46}$$

We know from Cauchy-Schwarz inequality that

$$\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n} \geq 0,$$

with equality if, and only if, ∇u is a conformal vector field on Σ. On the other hand, from our assumption (4.39) we also get

$$\begin{aligned}&\mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2 \\ &\quad \geq (K - (n-1))\|\nabla u\|^2 \geq 0\end{aligned}$$

Therefore, from (4.46) we conclude that

$$\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n} = 0 \tag{4.47}$$

and

$$\begin{aligned}&\mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2 \\ &\quad = (K - (n-1))\|\nabla u\|^2 = 0.\end{aligned} \tag{4.48}$$

Since $K > (n-1)$, (4.48) implies that u is constant and, by (4.40) it must be $u \equiv 1$. □

Remark 4.31. When $K = n-1$, if u is non-constant we conclude from (4.47) and (4.48) that ∇u is a conformal vector field on Σ which is a direction of least Ricci curvature at points where $\nabla u(p) \neq 0$. This is in fact what happens with the solutions given in Example 4.16, where

$$u(p) = f(p) = \frac{1}{\langle p, \mathbf{b}\rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

and $\Sigma = \mathbb{S}^n$ with the metric $\langle , \rangle = f^2 \langle , \rangle_0$.

4.6 Complete, non-compact, weakly trapped submanifolds through the light cone

In this section we consider the more general case of codimension two weakly trapped spacelike submanifolds which factorize through Λ^+. Recall that

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta$$

with

$$\theta_\xi = 1 \quad \text{and} \quad \theta_\eta = \frac{1}{nu}\Delta u - \frac{1 + \|\nabla u\|^2 - u^2}{2u^2},$$

where u is the positive function $u = -\langle \psi, \mathbf{e}_0 \rangle$. Therefore, Σ is (necessarily past) weakly trapped if, and only if, $\theta_\eta \geq 0$, that is,

$$2u\Delta u \geq n(1 + \|\nabla u\|^2 - u^2), \quad u > 0, \quad \text{on} \quad (\Sigma, \langle , \rangle). \tag{4.49}$$

Considering $v = u^2$ this is equivalent to

$$v\Delta v + nv^2 - nv \geq \frac{n+2}{4}\|\nabla v\|^2, \quad v > 0, \quad \text{on} \quad (\Sigma, \langle , \rangle).$$

In order to prove our result we need the following analytical theorem whose proof will be obtained as a modification of the proof of Theorem 3.3 in [37]. In what follows $r(p)$ denotes the Riemannian distance function from a reference point o of Σ.

Theorem 4.32. Let $(\Sigma, \langle , \rangle)$ be a complete, non-compact, Riemannian manifold and let $v > 0$ be a solution of

$$v\Delta v + av^2 - bv \geq -A\|\nabla v\|^2 \tag{4.50}$$

on Σ, with $a, b > 0$ and $A \in \mathbb{R}$. Suppose that φ is a positive C^2-solution of the differential inequality

$$\Delta \varphi + Ba\varphi \leq -C\frac{\|\nabla \varphi\|^2}{\varphi} \tag{4.51}$$

on Σ, and assume

$$A \leq B(C+1) - 1, \quad C > -1, \quad B > 0. \tag{4.52}$$

Let β satisfy

$$\beta > -1, \quad A \leq \beta \leq B(C+1) - 1. \tag{4.53}$$

Then

$$\left(\int_{\partial B_r} v^{2(\beta+1)}\right)^{-1} \in L^1(+\infty), \tag{4.54}$$

where B_r is the geodesic ball of radius r centered at the origin $o \in \Sigma$.

Proof. Let $v > 0$ be a solution of (4.50). Fix $\varepsilon > 0$, let

$$\alpha = B^{-1}(\beta + 1) > 0$$

and set
$$w_\varepsilon = \varphi^{-\alpha}(v^2+\varepsilon)^{(\beta+1)/2}.$$
It follows from (4.50) and (4.51) that

$$\begin{aligned} w_\varepsilon \mathrm{div}(\varphi^{2\alpha}\nabla w_\varepsilon) \;\geq\; & \alpha(C-\alpha+1)(v^2+\varepsilon)^{\beta+1}\frac{\|\nabla\varphi\|^2}{\varphi^2} + (\beta+1)(v^2+\varepsilon)^\beta bv \\ & + a(v^2+\varepsilon)^{\beta+1}\left(\alpha B - (\beta+1)\frac{v^2}{v^2+\varepsilon}\right) \\ & + (\beta+1)(v^2+\varepsilon)^\beta\left(1-A+(\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2. \end{aligned}$$

Since $\alpha B = \beta+1$, we have

$$\begin{aligned} & a(v^2+\varepsilon)^{\beta+1}\left(\alpha B-(\beta+1)\frac{v^2}{v^2+\varepsilon}\right) \\ & \quad = a(v^2+\varepsilon)^{\beta+1}\left(\beta+1-(\beta+1)\frac{v^2}{v^2+\varepsilon}\right) \\ & \quad = a(v^2+\varepsilon)^{\beta+1}(\beta+1)\left(1-\frac{v^2}{v^2+\varepsilon}\right) \\ & \quad = a\varepsilon(v^2+\varepsilon)^\beta(\beta+1). \end{aligned} \tag{4.55}$$

On the other hand, from our assumptions,

$$\begin{aligned} \alpha(C-\alpha+1) &= B^{-1}(\beta+1)(C-B^{-1}(\beta+1)+1) \\ &= B^{-2}(\beta+1)(B(C+1)-(\beta+1)) \geq 0 \end{aligned} \tag{4.56}$$

and also
$$(\beta+1)(v^2+\varepsilon)^\beta bv > 0. \tag{4.57}$$
Thus, using (4.55), (4.56) and (4.57) we have

$$\begin{aligned} w_\varepsilon \mathrm{div}(\varphi^{2\alpha}\nabla w_\varepsilon) \geq & (\beta+1)(v^2+\varepsilon)^\beta \varepsilon a + \\ & (\beta+1)(v^2+\varepsilon)^\beta\left(1-A+(\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2 \\ = & (\beta+1)(v^2+\varepsilon)^\beta\left(\varepsilon a + \left(1-A+(\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right). \end{aligned} \tag{4.58}$$

Consider the vector field
$$Z = w_\varepsilon\varphi^{2\alpha}\nabla w_\varepsilon.$$

For fixed t and $\delta > 0$, let ψ_δ be the Lipschitz function defined as

$$\psi_\delta(p) = \begin{cases} 1 & \text{if} \quad r(p) \leq t \\ \frac{t+\delta-r(p)}{\delta} & \text{if} \quad t < r(p) < t+\delta \\ 0 & \text{if} \quad r(p) \geq t+\delta. \end{cases}$$

Then, by (4.58) and the definition of ψ_δ we compute

$$\begin{aligned} \operatorname{div}(\psi_\delta Z) &= \psi_\delta \operatorname{div}(Z) + \langle \nabla\psi_\delta, Z\rangle \\ &= \left(w_\varepsilon \operatorname{div}(\varphi^{2\alpha}\nabla w_\varepsilon) + \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2\right)\psi_\delta + \langle \nabla\psi_\delta, Z\rangle \\ &\geq (\beta+1)(v^2+\varepsilon)^\beta \left(\varepsilon a + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right)\psi_\delta \\ &+ \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2 \chi_{B_t} - \frac{1}{\delta}\langle \nabla r, Z\rangle \chi_{\bar{B}_{t+\delta}\setminus B_t}, \end{aligned}$$

where we have used $\nabla\psi_\delta = -\frac{1}{\delta}\nabla r \chi_{\bar{B}_{t+\delta}\setminus B_t}$.

Then, integrating and using the divergence theorem and Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int_{\bar{B}_{t+\delta}} (\beta+1)(v^2+\varepsilon)^\beta \left(\varepsilon a + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right)\psi_\delta + \\ \int_{B_t} \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2 \leq \frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} \|Z\|. \end{aligned} \tag{4.59}$$

By Hölder inequality the integral on the right-hand side is bounded above as follows

$$\begin{aligned} \frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} \|Z\| &= \frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} w_\varepsilon \varphi^{2\alpha}\|\nabla w_\varepsilon\| \\ &= \int_{\bar{B}_{t+\delta}\setminus B_t} \left(\frac{\varphi^\alpha}{\sqrt{\delta}} w_\varepsilon\right)\left(\frac{\varphi^\alpha}{\sqrt{\delta}}\|\nabla w_\varepsilon\|\right) \\ &\leq \left(\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} \varphi^{2\alpha} w_\varepsilon^2\right)^{\frac{1}{2}} \left(\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Using this in (4.59) and letting $\delta \to 0^+$ we have

$$\begin{aligned}\int_{B_t} (\beta+1)(v^2+\varepsilon)^\beta \left(\varepsilon a + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right) + \\ \int_{B_t} \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2 \leq \left(\int_{\partial B_t} \varphi^{2\alpha} w_\varepsilon^2\right)^{\frac{1}{2}} \left(\int_{\partial B_t} \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2\right)^{\frac{1}{2}},\end{aligned} \tag{4.60}$$

where we have used the co-area formula, that is,

$$\lim_{\delta\to 0^+} \frac{1}{\delta} \int_{\bar{B}_{t+\delta}\setminus B_t} \varphi^{2\alpha} w_\varepsilon^2 = \int_{\partial B_t} \varphi^{2\alpha} w_\varepsilon^2$$

and

$$\lim_{\delta\to 0^+} \frac{1}{\delta} \int_{\bar{B}_{t+\delta}\setminus B_t} \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2 = \int_{\partial B_t} \varphi^{2\alpha}\|\nabla w_\varepsilon\|^2.$$

As $\varepsilon \to 0$, we have $w_\varepsilon \to w_0 = \varphi^{-\alpha} v^{(\beta+1)}$. Therefore, using the dominated convergence theorem in (4.60) we get

$$\begin{aligned}(\beta+1)(\beta - A) \int_{B_t} v^{2\beta}\|\nabla v\|^2 + \int_{B_t} \varphi^{2\alpha}\|\nabla w_0\|^2 \\ \leq \left(\int_{\partial B_t} \varphi^{2\alpha} w_0^2\right)^{\frac{1}{2}} \left(\int_{\partial B_t} \varphi^{2\alpha}\|\nabla w_0\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{4.61}$$

Define now

$$h(t) = \int_{B_t} \varphi^{2\alpha}\|\nabla w_0\|^2,$$

then, by the co-area formula, h is Lipschitz and

$$h'(t) = \int_{\partial B_t} \varphi^{2\alpha}\|\nabla w_0\|^2.$$

From our assumptions on β and A, we know that

$$(\beta+1)(\beta - A) \int_{B_t} v^{2\beta}\|\nabla v\|^2 \geq 0$$

so, from (4.61) we have that

$$h(t) \leq \left(\int_{\partial B_t} \varphi^{2\alpha} w_0^2\right)^{\frac{1}{2}} h'(t)^{\frac{1}{2}}. \tag{4.62}$$

Our aim now is to show that $w_0 = \varphi^{-\alpha} v^{\beta+1}$ is not constant. If w_0 is constant, then we can write $v^{\beta+1} = k\varphi^{\alpha}$ where k is a positive constant, $k > 0$. Using $\alpha = B^{-1}(\beta+1)$ we have $v^{\beta+1} = k\varphi^{B^{-1}(\beta+1)}$ and then

$$\varphi = cv^B,$$

where $c > 0$ constant. We rewrite now equation (4.51) using the previous expression for φ and we obtain

$$\Delta v^B + Bav^B \leq -C\frac{\|\nabla v^B\|^2}{v^B}. \tag{4.63}$$

Computing ∇v^B and Δv^B we have

$$\nabla v^B = Bv^{B-1}\nabla v, \qquad \|\nabla v^B\|^2 = B^2 v^{2B-2}\|\nabla v\|^2,$$

and

$$\Delta v^B = Bv^{B-1}\Delta v + B(B-1)v^{B-2}\|\nabla v\|^2.$$

Then, equation (4.63) becomes

$$v\Delta v + (B-1)\|\nabla v\|^2 + av^2 \leq -CB\|\nabla v\|^2,$$

or equivalently

$$v\Delta v + av^2 \leq \|\nabla v\|^2(1 - B(C+1)).$$

From the assumption (4.52) we have $1 - B(C+1) \leq -A$, and then, we have obtain

$$-A\|\nabla v\|^2 \geq v\Delta v + av^2 > v\Delta v + av^2 - bv$$

which contradicts (4.50), and we conclude that $w_0 = \varphi^{-\alpha}v^{\beta+1}$ is not constant. This implies that there exists $R_0 \gg 1$ such that $h(t) > 0$ for every $t \geq R_0$. Then, dividing in (4.62) by $h(t)$ we have

$$1 \leq \frac{h'(t)}{h(t)^2}\left(\int_{\partial B_t} \varphi^{2\alpha} w_0^2\right)$$

or, equivalently,

$$\frac{h'(t)}{h(t)^2} \geq \left(\int_{\partial B_t} \varphi^{2\alpha} w_0^2\right)^{-1}.$$

Taking $R_0 \leq r < R$ and integrating the previous inequality, it satisfies

$$\left(\int_{B_r} \varphi^{2\alpha} \|\nabla w_0\|^2\right)^{-1} = \frac{1}{h(r)} \geq \frac{1}{h(r)} - \frac{1}{h(R)}$$
$$= \int_r^R \frac{h'(t)}{h(t)^2} \geq \int_r^R \left(\int_{\partial B_t} \varphi^{2\alpha} w_0^2\right)^{-1}.$$

Taking into account that $\varphi^{2\alpha} w_0^2 = v^{2(\beta+1)}$ we have

$$\int_r^R \left(\int_{\partial B_t} v^{2(\beta+1)}\right)^{-1} \leq \left(\int_{B_r} \varphi^{2\alpha} \|\nabla w_0\|^2\right)^{-1} < +\infty$$

and from here we conclude that

$$\left(\int_{\partial B_t} v^{2(\beta+1)}\right)^{-1} \in L^1(+\infty).$$

□

Now we are ready to prove the next result for the non-compact case. Recall that if Σ is a codimension two complete, non-compact spacelike submanifold factorizing through Λ^+ we already know from Proposition 4.13 that u cannot be bounded above. Moreover,

$$\limsup_{r \to +\infty} \frac{u}{r \log(r)} = +\infty$$

where r is the Riemannian distance from a fixed origin $o \in \Sigma$. As a consequence of Theorem 4.32 we prove the following.

Theorem 4.33. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two complete, non-compact, weakly trapped spacelike submanifold which factorizes through Λ^+. Suppose that the first eigenvalue λ_1 of the Laplacian Δ is positive, that is

$$\lambda_1 = \inf_{\Omega} \lambda_1(\Omega) > 0$$

where Ω runs over all bounded domains of Σ. Let $u = -\langle \psi, \mathbf{e}_0 \rangle = \psi_0 > 0$. Then, for any q satisfying

$$0 < q \leq \frac{4}{n}\lambda_1, \tag{4.64}$$

we have

$$\left(\int_{\partial B_r} u^q\right)^{-1} \in L^1(+\infty). \tag{4.65}$$

In particular, $u \notin L^q(\Sigma)$.

Proof. We set $v = u^2$ to deduce from (4.49) the validity of

$$v\Delta v + nv^2 - nv \geq \frac{n+2}{4}\|\nabla v\|^2. \tag{4.66}$$

Since $\lambda_1 > 0$, the operator $L = \Delta + \lambda_1$ is non-negative, in the sense that

$$\lambda_1^L(\Omega) = \lambda_1(\Omega) - \lambda_1 \geq 0$$

for every bounded domain $\Omega \subset \Sigma$. Therefore, by a result of Fischer-Colbrie and Schoen [20] we deduce the existence of $\varphi \in \mathcal{C}^2(\Sigma)$, $\varphi > 0$, satisfying

$$\Delta\varphi + \lambda_1\varphi = 0 \quad \text{on} \quad \Sigma.$$

We now apply Theorem 4.32 with the choices $a = b = n$, $A = -(n+2)/4$, $B = \lambda_1/n$ and $C = 0$. Note that in this case assumption (4.52) is satisfied. Letting $\beta = q/4 - 1$, the validity of (4.64) gives the validity of (4.53). An application of Theorem 4.32 gives the desired conclusion. □

Corollary 4.34. Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ be a codimension two complete, weakly trapped spacelike submanifold which factorizes through Λ^+. Suppose that the first eigenvalue λ_1 of the Laplacian Δ is positive and let $u = -\langle \psi, \mathbf{e}_0 \rangle = \psi_0 > 0$. If $u \in L^q(\Sigma)$, for any q satisfying

$$0 < q \leq \frac{4}{n}\lambda_1,$$

then Σ is compact.

CHAPTER 5

A correspondence for codimension two spacelike submanifolds through a light cone

In this chapter we stablish a correspondence for codimension two spacelike submanifolds which factorize through a light cone of the $(n+2)$-dimensional de Sitter or anti-de Sitter spacetime and those which factorize through the light cone of the $(n+2)$-dimensional Lorentz-Minkowski spacetime. From now on, when we refer either to de Sitter or anti-de Sitter spacetime we will name it as (anti)-de Sitter spacetime and we will use the notation $\mathbb{M}_\varepsilon$, where $\varepsilon = 1$ for de Sitter spacetime and $\varepsilon = -1$ for anti-de Sitter spacetime. The study developed in this chapter is included in our work [13].

5.1 Preliminaries

First of all we should introduce anti-de Sitter spacetime, since we have already described de Sitter spacetime in Chapter 4. To do this, let us start by considering $\mathbb{R}^{n+3}_2$ the $(n+3)$-dimensional space given by the real vector space $\mathbb{R}^{n+3}$ endowed with the metric

$$\langle , \rangle = -(dx_0)^2 - (dx_1)^2 + (dx_2)^2 + \cdots + (dx_{n+2})^2,$$

where $(x_0, ..., x_{n+2})$ are the canonical coordinates of $\mathbb{R}^{n+3}$. In this setting we have the following standard definition for anti-de Sitter spacetime.

Definition 5.1. The $(n+2)$-dimensional anti-de Sitter spacetime is defined by the subset

$$\mathbb{H}^{n+2}_1 = \{x \in \mathbb{R}^{n+3}_2 : \langle x, x\rangle = -1\}$$

endowed with the induced metric from $\mathbb{R}_2^{n+3}$.

Now we intend to construct a globally defined timelike vector field $T \in \mathfrak{X}(\mathbb{H}_1^{n+2})$ in order to consider the time orientation on anti-de Sitter spacetime given by such T. With this aim in mind, for a fixed $\mathbf{a} \in \mathbb{H}_1^{n+2}$ let us consider $\mathbf{b} \in \mathbb{H}_1^{n+2}$ such that $\langle \mathbf{a}, \mathbf{b} \rangle = 0$ and we define $\widetilde{T} \in \mathfrak{X}(\mathbb{H}_1^{n+2})$ by

$$\widetilde{T}(x) = \langle \mathbf{a}, x \rangle \mathbf{b} - \langle \mathbf{b}, x \rangle \mathbf{a}$$

for every $x \in \mathbb{H}_1^{n+2}$. Now we compute

$$\langle \widetilde{T}(x), \widetilde{T}(x) \rangle = -\langle \mathbf{a}, x \rangle^2 - \langle \mathbf{b}, x \rangle^2 \leq -1 < 0.$$

Notice that every $x \in \mathbb{R}_2^{n+2}$ can be decomposed as

$$x = -\langle \mathbf{a}, x \rangle \mathbf{a} - \langle \mathbf{b}, x \rangle \mathbf{b} + x^*$$

where x^* is spacelike. Then, if we take $x \in \mathbb{H}_1^{n+2}$, it follows

$$-1 = \langle x, x \rangle = -\langle \mathbf{a}, x \rangle^2 - \langle \mathbf{b}, x \rangle^2 + \|x^*\|^2$$

and, since $\|x^*\|^2 \geq 0$, we have that $\langle \mathbf{a}, x \rangle^2 + \langle \mathbf{b}, x \rangle^2 = 1 + \|x^*\|^2 \geq 1 > 0$ for every $x \in \mathbb{H}_1^{n+2}$. At this point, we have obtained that we can consider on $\mathbb{H}_1^{n+2}$ the time orientation given by

$$T(x) = \frac{\widetilde{T}(x)}{|\widetilde{T}(x)|} = \frac{\langle \mathbf{a}, x \rangle \mathbf{b} - \langle \mathbf{b}, x \rangle \mathbf{a}}{\sqrt{\langle \mathbf{a}, x \rangle^2 + \langle \mathbf{b}, x \rangle^2}}, \tag{5.1}$$

which is a globally defined unit timelike vector field on $\mathbb{H}_1^{n+2}$.

As well as we denote by $\mathbb{M}_\varepsilon$ both de Sitter and anti-de Sitter spacetime, from here on we will use the notation $\mathbb{E}^{n+3}$ when we are referring to the $(n+3)$-dimensional Lorentz-Minkowski spacetime, $\mathbb{L}^{n+3}$, or to the $(n+3)$-dimensional real vector space of index 2, $\mathbb{R}_2^{n+3}$. In this way we have $\mathbb{M}_\varepsilon^{n+2} \subset \mathbb{E}^{n+3}$. Now, we present the notion of light cone in $\mathbb{M}_\varepsilon$.

Definition 5.2. Let $\mathbf{a} \in \mathbb{M}_\varepsilon$ be a fixed point of (anti)-de Sitter spacetime. The light cone of $\mathbb{M}_\varepsilon$ with vertex at $\mathbf{a}$ is the subset

$$\widetilde{\Lambda}_{\mathbf{a}} = \{x \in \mathbb{M}_\varepsilon : \langle x - \mathbf{a}, x - \mathbf{a} \rangle = 0, x \neq \mathbf{a}\},$$

which corresponds to the set of all the vectors $x \in \mathbb{M}_\varepsilon$ such that $x - \mathbf{a}$ is null.

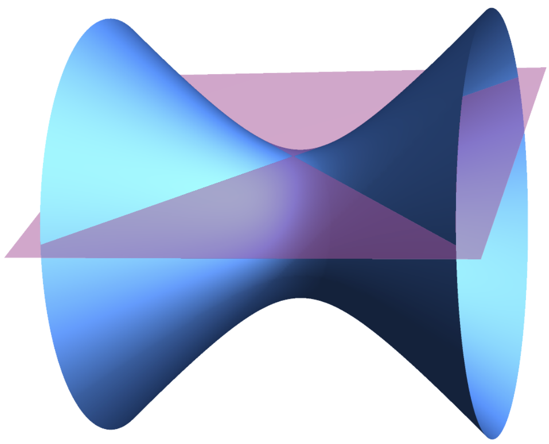

Figure 5.1: Light cone of anti-de Sitter spacetime

Remark 5.3. Recall that, from Remark 4.3 this definition of the light cone for the case of de Sitter spacetime is equivalent to the one given in Definition 5.2. On the other hand, for anti-de Sitter spacetime it is satisfied $\widetilde{\Lambda}_{\mathbf{a}} = \{x \in \mathbb{H}_1^{n+2} : \langle x, \mathbf{a}\rangle = -1, x \neq \mathbf{a}\}$. Then, in the general case of $\mathbb{M}_\varepsilon$, Definition 5.2 is equivalent to

$$\widetilde{\Lambda}_{\mathbf{a}} = \{x \in \mathbb{M}_\varepsilon : \langle \mathbf{a}, x\rangle = \varepsilon, x \neq \mathbf{a}\}.$$

5.2 Light cones in (anti)-de Sitter and Lorentz-Minkowski spacetimes

As we have already mentioned, our main goal here is to stablish a correspondence between codimension two submanifolds through a light cone of (anti)-de Sitter spacetime and those which factorize through the light cone of the Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$.

Let $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2}$ be a codimension two spacelike submanifold which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$ of (anti)-de Sitter $(n+2)$-dimensional spacetime $\mathbb{M}_\varepsilon$. Every such immersion determines the Lorentz-Minkowski space $\mathbf{a}^\perp \subset \mathbb{E}^{n+3}$ and so the light cone in $\mathbf{a}^\perp \simeq \mathbb{L}^{n+2}$ with vertex at the origin

$$\Lambda = \{x \in \mathbf{a}^\perp : \langle x, x\rangle = 0, x \neq \mathbf{0}\}.$$

Then, the translation

$$\begin{aligned} F : \mathbb{M}_\varepsilon^{n+2} &\to \mathbb{E}^{n+3} \\ x &\to x - \mathbf{a} \end{aligned} \tag{5.2}$$

induces an isometry from $\widetilde{\Lambda}_{\mathbf{a}}$ to Λ. By means of this isometry we have a one-to one correspondence between codimension two spacelike immersions $\psi : \Sigma^n \to \mathbb{M}_\varepsilon^{n+2}$ through $\widetilde{\Lambda}_{\mathbf{a}}$ and codimension two spacelike immersions $\phi : \Sigma^n \to \mathbb{L}^{n+2}$ through Λ. This correspondence can be summarized in the following proposition.

Proposition 5.4. Let $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2}$ be a codimension two spacelike submanifold which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. Then, there exists a spacelike immersion $\phi : \Sigma^n \to \Lambda \subset \mathbf{a}^\perp$ such that Σ factorizes through the light cone Λ and that makes commutative the following diagram,

$$\begin{tikzcd} \Lambda \subset \mathbf{a}^\perp \arrow[r, "j"] & \mathbb{E}^{n+3} \\ \Sigma^n \arrow[u, "\phi"] \arrow[r, "\psi"'] & \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2} \arrow[u, "F"'] \end{tikzcd} \tag{5.3}$$

where $F(x) = x - \mathbf{a}$ induces a (totally umbilical) isometry from $\widetilde{\Lambda}_{\mathbf{a}}$ to Λ and j is the totally geodesic inclusion. In particular, the intrinsic geometries on Σ induced by ψ and ϕ are the same.

Remark 5.5. As a consequence of [18, Corollary 7.6] (see also [9]) and the previous proposition, we can deduce that any Riemannian manifold M^n, $n \geq 3$, is locally conformally flat if, and only if, it can be locally isometrically immersed in the light cone $\widetilde{\Lambda}_{\mathbf{a}}$.

We fix now $\mathbf{w} \in \mathbf{a}^\perp$ a unit timelike vector. We can consider the future light cone $\Lambda^+ \subset \mathbf{a}^\perp$ and the past light cone $\Lambda^- \subset \mathbf{a}^\perp$ with repect to this vector $\mathbf{w}$. Observe that, using the isometry F defined in Proposition 5.4, we are able to define the future and the past components of the light cone $\widetilde{\Lambda}_{\mathbf{a}}$ with respect to $\mathbf{w}$ in natural way. As usual in previous chapters, since Σ is always conected, we can suppose $\psi(\Sigma) \subset \widetilde{\Lambda}_{\mathbf{a}}^+$.

Once we have fixed $\mathbf{w} \in \mathbf{a}^\perp$ we can also define the height function on Σ as

$$\begin{aligned} h_{\mathbf{w}} : \Sigma^n &\to \mathbb{R} \\ x &\to -\langle \phi(x), \mathbf{w} \rangle = -\langle \psi(x), \mathbf{w} \rangle. \end{aligned} \tag{5.4}$$

Note that $h_{\mathbf{w}}(x) \neq 0$ for every $x \in \Sigma$ and, if $\phi(\Sigma) \subset \Lambda^+$, then $h_{\mathbf{w}} > 0$.

At this point we wonder what we can say about the extrinsic geometries of ψ and ϕ. Let us denote by $\mathbf{H}$ with a subscript the mean curvature vector field corresponding to each given immersion. With this notation we can state next result.

Proposition 5.6. Let $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2}$ be a codimension two spacelike submanifold which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. Then

$$\langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle = \langle \mathbf{H}_\phi, \mathbf{H}_\phi \rangle - \varepsilon, \tag{5.5}$$

where $\phi : \Sigma^n \to \Lambda \subset \mathbf{a}^\perp$ is the corresponding immersion given in Proposition 5.4.

Proof. By the commutative diagram (5.3) we have $\mathbf{H}_{j\circ\phi} = \mathbf{H}_{T\circ\psi}$, and, since T is totally umbilical, it follows $\mathbf{H}_{T\circ\psi} = \mathbf{H}_\psi + \varepsilon\zeta$ where ζ is the outward unit normal vector field to $T : \mathbb{M}_\varepsilon \to \mathbb{E}^{n+3}$, $\langle \zeta, \zeta \rangle = \varepsilon$. Finally, taking into account that j is a totally geodesic immersion we obtain $\langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle = \langle \mathbf{H}_\phi, \mathbf{H}_\phi \rangle - \varepsilon$, as we wanted to prove. $\square$

In the next corollary, as a direct consequence of equation (3.5) and previous proposition, we obtain an expression for the scalar curvature of Σ.

Corollary 5.7. Let $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2}$ be a codimension two spacelike submanifold which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. Then, the scalar curvature of Σ is given by

$$\mathrm{Scal} = n(n-1)(\langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle + \varepsilon).$$

On the other hand, by the commutative diagram (5.3) we know that every codimension two spacelike submanifold $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2}$ is totally umbilical if, and only if, $\phi : \Sigma^n \to \Lambda \subset \mathbf{a}^\perp$ is totally umbilical. Then, the following result is a direct consequence of Theorem 3.9.

Proposition 5.8. Let $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}_\varepsilon^{n+2}$ be a codimension two spacelike submanifold which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. If ψ is totally umbilical, then there exist $\mathbf{v} \in \mathbb{E}^{n+3}$ and $\tau > 0$ such that

$$\psi(\Sigma) \subset \Sigma(\mathbf{v}, \tau) = \{x \in \widetilde{\Lambda}_{\mathbf{a}} : \langle x - \mathbf{a}, \mathbf{v} \rangle = \tau\}.$$

Finally in this section we obtain a result for the compact case which follows from Proposition 3.12 and that, when $\varepsilon = 1$, it is nothing but Proposition 4.13.

Proposition 5.9. Let $\psi : \Sigma^n \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^{n+2}_{\varepsilon}$ be a codimension two compact spacelike submanifold factorizing through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. If the height function $h_{\mathbf{w}}$ defining in (5.4) is bounded from above, then Σ is conformally diffeomorphic to the Euclidean sphere $\mathbb{S}^n$.

Remark 5.10. Actually, in Proposition 5.9 it is enough to assume that $h_{\mathbf{w}}$ satisfies condition (4.19), as we did in Proposition 3.12 and Proposition 4.13.

5.3 Spacelike surfaces through a light cone of (anti)-de Sitter spacetime

In this section we focus on the case $n = 2$, that is, Σ is now a spacelike surface which factorizes through $\widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_{\varepsilon}$. From Corollary 5.7 we immediately obtain next identity.

Corollary 5.11. Let $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_{\varepsilon}$ be a spacelike surface with factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. Then, the Gauss curvature of Σ can be written as

$$K = \langle \mathbf{H}_{\psi}, \mathbf{H}_{\psi} \rangle + \varepsilon.$$

On the other hand, taking into account the definition of the height function $h_{\mathbf{w}}$ in (5.4) and [45, Corollary 3.7] we obtain the following expression for the Gauss curvature of Σ.

Corollary 5.12. Let $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_{\varepsilon}$ be a spacelike surface which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. Then, for every unit timelike vector $\mathbf{w} \in \mathbf{a}^{\perp}$, the Gauss curvature of Σ is given by

$$K = \frac{1 + |\nabla h_{\mathbf{w}}|^2}{h_{\mathbf{w}}^2} - \frac{\Delta h_{\mathbf{w}}}{h_{\mathbf{w}}}. \tag{5.6}$$

In particular, when ϕ factorizes through the future light cone $\Lambda^+ \subset \mathbf{a}^{\perp}$ with respect to $\mathbf{w}$, we have

$$K = \frac{1}{h_{\mathbf{w}}^2} - \Delta \log(h_{\mathbf{w}}).$$

Remark 5.13. For example, the vector $P = -\varepsilon \mathbf{e}_0 + \langle \mathbf{e}_0, \mathbf{a} \rangle \mathbf{a}$ satisfies $P \in \mathbf{a}^\perp$ and it is not difficult to show that P is timelike in de Sitter case ($\varepsilon = 1$). Then, the height function $h_{\mathbf{w}}$ for $\mathbf{w} := \frac{1}{\sqrt{-\langle P,P \rangle}} P$ is given by,

$$h_{\mathbf{w}}(x) = \frac{1}{\sqrt{1 + \varepsilon \langle \mathbf{e}_0, \mathbf{a} \rangle^2}} \langle \psi(x) - \mathbf{a}, \mathbf{e}_0 \rangle,$$

for every $x \in \Sigma$.

Now we can relate the sign of the Gauss curvature K with the existence of local extreme points of the function $h_{\mathbf{w}}$ using [45, Proposition 3.11] as follows.

Proposition 5.14. Let $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_\varepsilon$ be a spacelike surface that factorizes through the future (resp. past) component of the light cone $\Lambda_{\mathbf{a}}$ with respect to $\mathbf{w} \in \mathbf{a}^\perp$ and with Gauss curvature $K \leq 0$. Then, the function $h_{\mathbf{w}}$ does not attain a local maximum (resp. minimum) value.

5.3.1 Compact spacelike surfaces through a light cone of (anti)-de Sitter spacetime

This part of the section is devoted to the case when the spacelike surface Σ which factorizes throguh the light cone of (anti)-de Sitter spacetime is compact. In this instance, and even irrespective of Proposition 5.8, we can characterize the totally umbilical surfaces in $\mathbb{M}_\varepsilon$ through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$ using Proposition 5.9 and [45, Theorem 5.4].

Proposition 5.15. Let $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_\varepsilon$ be a compact spacelike surface which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. If K is constant, then Σ is a totally umbilical round sphere.

As a direct application of previous proposition, we have the following.

Corollary 5.16. Let $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_\varepsilon$ be a compact spacelike surface which factorizes through the light cone $\widetilde{\Lambda}_{\mathbf{a}}$. The following assertions are equivalent:

(i) K is constant.

(ii) ψ is totally umbilical.

(iii) There exist $\tau > 0$ and $\mathbf{v} \in \mathbf{a}^\perp$ with $\langle \mathbf{v}, \mathbf{v} \rangle = -1$ such that

$$\psi(\Sigma) = \Sigma(\mathbf{v}, \tau) = \{x \in \widetilde{\Lambda}_{\mathbf{a}} : \langle x, \mathbf{v} \rangle = \tau\}.$$

Taking now into account Proposition 5.9, Corollary 5.11 and the Gauss-Bonnet formula, the total mean curvature of compact spacelike surfaces through the light cone $\Lambda_{\mathbf{a}} \subset \mathbb{M}_\varepsilon$ may be expressed as

$$\int_\Sigma \langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle dA = 4\pi - \varepsilon\, \mathsf{Area}(\Sigma). \tag{5.7}$$

Formula (5.7) shows that, for compact spacelike surfaces that factorize through a light cone in $\mathbb{M}^4_\varepsilon$, the identity in [51, Theorem 1] holds.

In order to analyse the spectrum of the Laplacian operator of $(\Sigma, \langle , \rangle)$, formula (5.7) proves to be very useful. First, let us recall that for an arbitrary Riemannian metric g on $\mathbb{S}^2$, the minimum non zero eigenvalue of the Laplacian operator λ_1 of g satisfies Hersch inequality [27], which states

$$\lambda_1 \le \frac{8\pi}{\mathsf{Area}(\mathbb{S}^2, g)},$$

and the equality holds if, and only if, $(\mathbb{S}^2, g)$ has constant Gauss curvature. Therefore, taking into account formula (5.7), Hersch inequality may be written for a compact spacelike surface Σ through $\widetilde{\Lambda}_{\mathbf{a}}$ as

$$\lambda_1 \le \frac{2\int_\Sigma \langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle dA}{\mathsf{Area}(\Sigma)} + 2\varepsilon, \tag{5.8}$$

and the equality holds if, and only if, $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_\varepsilon$ is totally umbilical. This formula gives an extrinsic bound of the first non trivial eigenvalue of the Laplacian operator of $(\Sigma, \langle , \rangle)$, and, in some sense, it corresponds to the well-known Reilly's inequality in the Euclidean space, although it is known that Reilly's inequality is not true in general in a Lorentzian ambient (see for instance [45]).

In the compact case, Corollary 5.12 gives the following integral formula for a compact spacelike surface Σ through $\widetilde{\Lambda}_{\mathbf{a}}$,

$$\int_\Sigma \frac{1}{h_{\mathbf{w}}^2} dA = 4\pi, \tag{5.9}$$

for every unit timelike vector $\mathbf{w} \in \mathbf{a}^\perp$. Now, from Schwarz inequality and Theorem 5.8 we come to the following result.

Proposition 5.17. Let $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_{\varepsilon}$ be a compact spacelike surface that factorizes through the future component of the light cone $\widetilde{\Lambda}_{\mathbf{a}}$ with respect to the unit timelike vector $\mathbf{w} \in \mathbf{a}^{\perp}$. Then, we have the following upper bound for the area of Σ,

$$\text{Area}(\Sigma) \leq 2\sqrt{\pi}\|-h_{\mathbf{w}}\|,$$

where $\|\cdot\|$ is the usual L^2 norm. Moreover, the equality holds for some $\mathbf{w}$ if, and only if, Σ is the totally umbilical round sphere $\Sigma(\mathbf{w}, r)$ with $r = 1/h_{\mathbf{w}}$.

Finally, from formula (5.7), Hersch inequality and Corollary 5.16 we get the next theorem.

Theorem 5.18. Let $\psi : \Sigma \to \widetilde{\Lambda}_{\mathbf{a}} \subset \mathbb{M}^4_{\varepsilon}$ be a compact spacelike surface which factorizes through the future component of the light cone $\widetilde{\Lambda}_{\mathbf{a}}$ with respect to the unit timelike vector $\mathbf{w} \in \mathbf{a}^{\perp}$. Then, for every unit timelike vector $\widetilde{\mathbf{w}} \in \mathbf{a}^{\perp}$ with $\langle \mathbf{w}, \widetilde{\mathbf{w}} \rangle < 0$, we have

$$\lambda_1 \leq \frac{2}{\min_{\Sigma}(h^2_{\widetilde{\mathbf{w}}})},$$

and the equality holds for some $\widetilde{\mathbf{w}}$ if, and only if, Σ is immersed as a totally umbilical round sphere in $\mathbb{M}^4_{\varepsilon}$.

$\square$

CHAPTER 6

Trapped submanifolds in generalized Robertson-Walker spacetimes

In this chapter, and following the terminology introduced in [7], our ambient space will be an $(n+2)$-dimensional generalized Robertson-Walker (GRW) spacetime with $n \geq 2$. In this case our research is developed in the case of codimension two spacelike submanifolds which are trapped, marginally trapped or weakly trapped. The content shown here corresponds essentially to that of our publication [3].

6.1 Preliminaries

As we shall see, GRW spaces are part of a bigger family, the family of the well known warped products. Let us start by defining a warped product of arbitrary dimension in order to introduce the GRW spacetimes later.

Definition 6.1. Let $(B, \langle,\rangle_B)$ and $(N, \langle,\rangle_N)$ be two semi-Riemannian manifolds and let $\varrho : B \to (0, +\infty)$ be a smooth positive function called warping function. The warped product denoted by $M = B \times_\varrho N$ is the product manifold $B \times N$ endowed with the metric

$$\langle,\rangle = \pi_B^*(\langle,\rangle_B) + (\varrho \circ \pi_B)^2 \pi_N^*(\langle,\rangle_N),$$

where π_B and π_N stands for the canonical projections onto B and N respectively.

Observe that when ϱ is the constant function $\varrho = 1$ then $\langle,\rangle$ is nothing but the metric of the product manifold $B \times N$. For simplicity, in what follows we are going to write the warped metric as

$$\langle , \rangle = \langle , \rangle_B + \varrho^2 \langle , \rangle_N.$$

Following the usual notation, if $B \times_\varrho N$ is a warped product, we will call B its basis and N its fiber. In this way, any vector field Z on a warped product $B \times_\varrho N$ can be decomposed as

$$Z = Z^* + Z_* \tag{6.1}$$

where Z^* is the component tangent to the fiber, $Z^* \in \mathfrak{X}(N)$, and Z_* is the component tangent to the basis, $Z_* \in \mathfrak{X}(B)$.

Next proposition shows how the Levi-Civita connection of a warped product acts. Here the sets of all lifts on B and N are denoted respectively as $\mathcal{L}(B)$ and $\mathcal{L}(N)$. Typically we use the same notation for a vector field and for its lift, as we have already done in formula (6.1).

Proposition 6.2. Let $B \times_\varrho N$ be a warped product. Let $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(N)$ be four vector fields. Then,

(i) $\overline{\nabla}_X Y = \nabla^*_X Y$, where ∇^* denotes the Levi-Civita connection of B.

(ii) $\overline{\nabla}_X V = \overline{\nabla}_V X = \frac{X(\varrho)}{\varrho} V$.

(iii) $(\overline{\nabla}_V W)_* = \mathrm{II}(V, W) = -\dfrac{\langle V, W \rangle}{\varrho} \nabla \varrho$.

(iv) $(\overline{\nabla}_V W)^* = \widetilde{\nabla}_V W$ where $\widetilde{\nabla}$ is the Levi-Civita connection of N.

Now we have the tools to start our study in a GRW spacetime, which we present below.

Definition 6.3. Let $(N, \langle , \rangle_N)$ be a Riemannian manifold and I an open interval of $\mathbb{R}$. The generalized Robertson-Walker (GRW) spacetime denoted by $-I \times_\varrho N^{n+1}$ is the product manifold $I \times N$ endowed with the Lorentzian warped metric

$$\langle , \rangle = -dt^2 + \varrho^2 \langle , \rangle_N, \tag{6.2}$$

where $\varrho : I \longrightarrow (0, +\infty)$ is a positive smooth function on I.

In other words, $-I \times_\varrho N^{n+1}$ is nothing but a Lorentzian warped product with Lorentzian base $(I, -dt^2)$, Riemannian fiber $(N^{n+1}, \langle , \rangle_N)$, and warping function ϱ. As usual, when the fiber has constant curvature we will refer to $-I \times_\varrho N^{n+1}$ as a Robertson-Walker (RW) spacetime .

We will choose the time orientation on $-I \times_{\varrho} N^{n+1}$ given by the globally defined timelike unit vector field

$$\partial_t = (\partial/\partial t)_{|_{(t,x)}}, \quad (t,x) \in -I \times_{\varrho} N^{n+1}.$$

If we take Z a vector field on $-I \times_{\varrho} N^{n+1}$, we can decompose it as in formula (6.1), $Z = Z^* + Z_*$, with $Z^* \in \mathfrak{X}(N)$ and $Z_* \in \mathfrak{X}(I)$. Since $\mathfrak{X}(I)$ has dimension one and $\partial_t \in \mathfrak{X}(I)$ we have that

$$Z_* = \lambda \partial_t$$

for a certain funcion $\lambda \in \mathcal{C}^\infty(-I \times_{\varrho} N^{n+1})$. From this identity we obtain that $\langle Z_*, \partial_t \rangle = -\lambda$ and hence, $\lambda = -\langle Z_*, \partial_t \rangle = -\langle Z, \partial_t \rangle$. Therefore, every vector field $Z \in \mathfrak{X}(-I \times_{\varrho} N^{n+1})$ can be written as

$$Z = Z^* - \langle Z, \partial_t \rangle \partial_t \tag{6.3}$$

where $Z^* \in \mathfrak{X}(N)$.

Let us now consider the vector field given by

$$K(t,x) = f(t)(\partial/\partial t)_{(t,x)}, \quad (t,x) \in -I \times_{\varrho} N^{n+1}. \tag{6.4}$$

It determines a non-vanishing future-pointing conformal vector field on $-I \times_{\varrho} N^{n+1}$ which is also closed, in the sense that its metrically equivalent 1-form is closed. In fact, for every vector field $Z \in \mathfrak{X}(-I \times_{\varrho} N^{n+1})$,

$$\begin{aligned}
\overline{\nabla}_Z K &= \overline{\nabla}_{Z^*} K - \langle Z, \partial_t \rangle \overline{\nabla}_{\partial_t} K \\
&= K(\varrho)\varrho Z^* - \langle Z, \partial_t \rangle \varrho' \partial_t \\
&= \varrho'(Z^* - \langle Z, \partial_t \rangle \partial_t) \\
&= \varrho' Z,
\end{aligned} \tag{6.5}$$

where we are using Proposition 6.2 and the fact that $\overline{\nabla}_{\partial_t}\partial_t = 0$. In particular, GRW spacetimes are conformally stationary spacetimes. This conformal field K will be an essential tool in our subsequent computations.

As we will see later, the hypothesis $(\log \varrho)'' \leq 0$ will be assumed in some of the results that we present in this chapter. To end this subsection, let us dedicate a few words concerning this condition and its relation with the so called timelike convergence condition (TCC). Recall that a spacetime obeys the TCC if its Ricci curvature is non-negative on timelike directions. It is not difficult to see that a

GRW spacetime $-I \times_{\varrho} N^{n+1}$ obeys TCC if, and only if,

$$\mathrm{Ric}_N \geq n \sup_{t\in I}(\varrho\varrho'' - (\varrho')^2)\langle,\rangle_N = n \sup_{t\in I}(-\mathcal{H}'(t)\varrho(t)^2)\langle,\rangle_N \tag{6.6}$$

and

$$\varrho''(t) \leq 0, \tag{6.7}$$

where Ric_N and $\langle,\rangle_N$ are respectively the Ricci and metric tensors of the Riemannian fiber N^{n+1}. In particular, the sole condition (6.7) implies $(\log \varrho)'' \leq 0$, which is enough, in many cases, to guarantee rigidity of constant (higher order) mean curvature spacelike hypersurfaces (see for instance [6], bearing in mind that the convention for the second fundamental form is the opposite to ours).

6.1.1 Slices in GRW spacetimes

Now we focus our study on the slices, a distinguished family of hypersurfaces of a GRW spacetime which we start introducing in the definition below. Concretely we will compute the induced metric, the shape operator and the mean curvature of such a hypersurface.

Definition 6.4. Let $-I \times_{\varrho} N^{n+1}$ be a GRW spacetime. For a fixed $\tau \in I$, the slice N_τ is the hypersurface $\{\tau\} \times N \subset -I \times_{\varrho} N^{n+1}$.

Any slice N_τ of a GRW spacetime is an embedded spacelike hypersurface, in the sense that the induced metric on N_τ from the Lorentzian metric (6.2) is Riemannian. Actually, the induced metric on N_τ is given by $\varrho(\tau)^2\langle,\rangle_N$, that is, N_τ is homothetic to N with scale factor $\varrho(\tau)$.

The restriction of ∂_t to N_τ gives its future-directed Gauss map and it follows from (6.5) that

$$\overline{\nabla}_V \partial_t = \overline{\nabla}_V \left(\frac{1}{\varrho(t)}K\right) = -\frac{1}{\varrho(t)^2}\langle V, \overline{\nabla}\bar{\varrho}\rangle K + \frac{1}{\varrho(t)}\varrho'(t)V \tag{6.8}$$

for every vector field V on $-I \times_{\varrho} N^{n+1}$, where $\overline{\nabla}\bar{\varrho}$ denotes the gradient of $\bar{\varrho}(t,x) = \varrho(t)$ on $-I \times_{\varrho} N^{n+1}$.

Observe that the gradient on $-I \times_{\varrho} N^{n+1}$ of the projection $\pi_I(t,x) = t$ is given by

$$\overline{\nabla}\pi_I = -\langle\overline{\nabla}\pi_I, \partial_t\rangle\partial_t = -\partial_t. \tag{6.9}$$

Thus, writing $\bar{\varrho} = \varrho \circ \pi_I$ we get

$$\overline{\nabla}\bar{\varrho} = \varrho'(t)\overline{\nabla}\pi_I = -\varrho'(t)\partial_t$$

and (6.8) becomes

$$\overline{\nabla}_V \partial_t = \frac{\varrho'(t)}{\varrho(t)}(V + \langle V, \partial_t\rangle \partial_t), \tag{6.10}$$

for every vector field V on $-I \times_\varrho N^{n+1}$. In particular,

$$\overline{\nabla}_{\mathbf{v}} \partial_t = \frac{\varrho'(\tau)}{\varrho(\tau)}\mathbf{v}$$

for every tangent vector $\mathbf{v} \in T_{(\tau,x)}N_\tau$. This means that N_τ is a totally umbilical hypersurface in $-I \times_\varrho N^{n+1}$ with shape operator (with respect to the future-directed Gauss map ∂_t) given by

$$A_\tau \mathbf{v} = \overline{\nabla}_{\mathbf{v}} \partial_t = \frac{\varrho'(\tau)}{\varrho(\tau)}\mathbf{v}$$

for every $\mathbf{v} \in T_{(\tau,x)}N_\tau$.

Therefore, $\tau \in I \to N_\tau \subset -I \times_\varrho N^{n+1}$ determines a foliation of $-I \times_\varrho N^{n+1}$ by totally umbilical spacelike hypersurfaces with future constant mean curvature given by

$$\mathcal{H}(\tau) := -\frac{1}{n+1}\mathrm{tr}(A_\tau) = -\frac{\varrho'(\tau)}{\varrho(\tau)}.$$

The choice of the negative sign in our definition of the mean curvature is motivated by the fact that, with this approach, the mean curvature vector field of N_τ is given by $\mathcal{H}(\tau)\partial_t|_{(\tau,x)}$. Therefore, $\mathcal{H}(\tau) > 0$ (that is, $\varrho'(\tau) < 0$) if, and only if, the mean curvature vector of N_τ is future-pointing.

6.2 Marginally trapped submanifolds contained in slices

In this section we will see the n-dimensional manifold Σ from two different points of view. On the one hand, we will consider it as an immersed hypersurface in the Riemannian manifold $(N^{n+1}, \langle,\rangle_N)$. In other words, there exists a smooth immersion $\phi : \Sigma \to N^{n+1}$. We will denote by $\langle,\rangle_\Sigma$ the Riemannian induced metric on Σ via ϕ, that is, $\langle,\rangle_\Sigma = \phi^*(\langle,\rangle_N)$.

On the other hand, Σ will be a spacelike codimension two submanifold on $-I\times_{\varrho}N^{n+1}$ which is contained in a slice N_τ, that is, $\psi:\Sigma\to -I\times_{\varrho}N^{n+1}$ is a spacelike immersion such that $\psi(\Sigma)\subset N_\tau$.

First, for a fixed $\tau\in I$, we define $\phi_\tau:\Sigma\to -I\times_{\varrho}N^{n+1}$ the map given by

$$\phi_\tau(p)=(\tau,\phi(p)),\qquad \text{for every } p\in\Sigma.$$

A straightforward computation shows that ϕ_τ is a spacelike immersion of Σ into $-I\times_{\varrho}N^{n+1}$ which is contained in the slice N_τ, and that the metric induced on Σ via ϕ_τ from the Lorentzian metric (6.2) is simply

$$\langle,\rangle_\tau=\phi_\tau^*(\langle,\rangle)=\varrho(\tau)^2\langle,\rangle_\Sigma. \tag{6.11}$$

Conversely, it is not difficult to see that the projection $\phi=\pi_N\circ\psi:\Sigma\to N^{n+1}$ is an immersed hypersurface for which

$$\psi(p)=(\tau,\phi(p))=\phi_\tau(p).$$

Moreover, $\psi:\Sigma\to -I\times_{\varrho}N^{n+1}$ is an embedding if, and only if, $\phi:\Sigma\to N^{n+1}$ is an embedding.

It follows from (6.11) that, intrinsically, $(\Sigma,\langle,\rangle_\tau)$ is homothetic to $(\Sigma,\langle,\rangle_\Sigma)$ with scale factor $\varrho(\tau)$. Our objective now is to express the extrinsic geometry of the codimension two spacelike submanifold $\phi_\tau:\Sigma\to -I\times_{\varrho}N^{n+1}$ in terms of the extrinsic geometry of the hypersurface $\phi:\Sigma\to N^{n+1}$. In order to compute the second fundamental form II_τ of the immersion ϕ_τ, let ζ denote the (locally defined) unit normal vector field of the hypersurface $\phi:\Sigma\to N^{n+1}$, with $\langle\zeta,\zeta\rangle_N=1$. Notice that

$$\langle\zeta,\zeta\rangle=\varrho(\tau)^2\langle\zeta,\zeta\rangle_N=\varrho(\tau)^2. \tag{6.12}$$

Then,

$$\xi_\tau(p)=\partial_t|_{(\tau,\phi(p))},\qquad\text{and}\qquad \eta_\tau(p)=\frac{1}{\varrho(\tau)}\zeta(p),\quad p\in\Sigma,$$

defines a local orthonormal frame of vector fields along the immersion ϕ_τ, with

$$\langle\xi_\tau,\xi_\tau\rangle=-1,\quad \langle\eta_\tau,\eta_\tau\rangle=1\quad\text{and}\quad\langle\eta_\tau,\xi_\tau\rangle=0.$$

The second fundamental form II_τ of the immersion ϕ_τ is then written as

$$\begin{aligned}\mathrm{II}_\tau(X,Y)&=\langle\mathrm{II}_\tau(X,Y),\eta_\tau\rangle\eta_\tau-\langle\mathrm{II}_\tau(X,Y),\xi_\tau\rangle\xi_\tau\\&=\langle A_{\eta_\tau}X,Y\rangle_\tau\eta_\tau-\langle A_{\xi_\tau}X,Y\rangle_\tau\xi_\tau,\end{aligned} \tag{6.13}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Observe that, for every $X \in \mathfrak{X}(\Sigma)$,

$$\overline{\nabla}_X \eta_\tau = \frac{1}{\varrho(\tau)} \overline{\nabla}_X \zeta = \frac{1}{\varrho(\tau)} \widetilde{\nabla}_X \zeta,$$

where $\widetilde{\nabla}$ denotes the Levi-Civita connection of $(N^{n+1}, \langle , \rangle_N)$. Therefore, taking into account that

$$AX = \widetilde{\nabla}_X \zeta,$$

where $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator of the hypersurface $\phi \colon \Sigma \to N^{n+1}$ with respect to ζ, it follows from here, using the Weingarten formula (2.5), that $\nabla^\perp_X \eta_\tau = 0$ and

$$A_{\eta_\tau} X = \overline{\nabla}_X \eta_\tau = \frac{1}{\varrho(\tau)} AX$$

for every $X \in \mathfrak{X}(\Sigma)$. On the other hand, since $\langle X, \xi_\tau \rangle = 0$ for every $X \in \mathfrak{X}(\Sigma)$, by (6.10) we have

$$\overline{\nabla}_X \xi_\tau = \frac{\varrho'(\tau)}{\varrho(\tau)} X,$$

which yields $\nabla^\perp_X \xi_\tau = 0$ and

$$A_{\xi_\tau} X = \frac{\varrho'(\tau)}{\varrho(\tau)} X$$

for every $X \in \mathfrak{X}(\Sigma)$. Putting this into (6.13) we obtain that

$$\mathrm{II}_\tau(X, Y) = \frac{1}{\varrho(\tau)^2} \langle AX, Y \rangle_\tau \zeta - \frac{\varrho'(\tau)}{\varrho(\tau)} \langle X, Y \rangle_\tau \xi_\tau, \tag{6.14}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Thus, the mean curvature vector field of ϕ_τ is given by

$$\begin{aligned}
\mathbf{H}_\tau &= \frac{1}{n} \mathrm{tr}(\mathrm{II}_\tau) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{II}_\tau(E_i, E_i) \\
&= \frac{1}{n} \left(\frac{1}{\varrho(\tau)^2} \sum_{i=1}^{n} \langle AE_i, E_i \rangle_\tau \zeta - \frac{\varrho'(\tau)}{\varrho(\tau)} \sum_{i=1}^{n} \langle E_i, E_i \rangle_\tau \xi_\tau \right) \\
&= \frac{1}{n} \left(\frac{1}{\varrho(\tau)^2} \sum_{i=1}^{n} \langle AE_i, E_i \rangle_\tau \zeta - n \frac{\varrho'(\tau)}{\varrho(\tau)} \xi_\tau \right)
\end{aligned} \tag{6.15}$$

where $\{E_1, \dots, E_n\}$ is a local orthonormal frame on Σ, with respect to the metric $\langle , \rangle_\tau$.

On the other hand, by (6.11) we have that

$$\langle AE_i, E_i \rangle_\tau = \varrho(\tau)^2 \langle AE_i, E_i \rangle_\Sigma = \langle Ae_i, e_i \rangle_\Sigma$$

for every $i = 1, \dots, n$, where $e_i = \varrho(\tau)E_i$ and $\{e_1, \dots, e_n\}$ is a local orthonormal frame on Σ with respect to the metric $\langle , \rangle_\Sigma$. Observe that the mean curvature function of the hypersurface $\phi : \Sigma \to N^{n+1}$ is given by

$$H = \frac{1}{n}\mathrm{tr}(A) = \frac{1}{n}\sum_{i=1}^{n} \langle Ae_i, e_i \rangle_\Sigma = \frac{1}{n}\sum_{i=1}^{n} \langle AE_i, E_i \rangle_\tau .$$

Putting this into (6.15) we conclude that

$$\mathbf{H}_\tau = \frac{H}{\varrho(\tau)^2}\zeta - \frac{\varrho'(\tau)}{\varrho(\tau)}\xi_\tau, \tag{6.16}$$

and, in particular, using (6.12), it follows

$$\langle \mathbf{H}_\tau, \mathbf{H}_\tau \rangle = \frac{H^2 - \varrho'(\tau)^2}{\varrho(\tau)^2}.$$

This means that ϕ_τ is a weakly trapped submanifold of $-I \times_\varrho N^{n+1}$ if, and only if, the mean curvature function H of the hypersurface $\phi : \Sigma \to N^{n+1}$ is bounded by

$$H^2 \leq \varrho'(\tau)^2.$$

Specifically, the immersion ϕ_τ is marginally trapped if, and only if, $\phi : \Sigma \to N^{n+1}$ is a hypersurface with constant mean curvature $H = \pm\varrho'(\tau) \neq 0$. Moreover,

$$\langle \mathbf{H}_\tau, \xi_\tau \rangle = \frac{\varrho'(\tau)}{\varrho(\tau)} = \text{constant}.$$

Therefore, when causal, $\mathbf{H}_\tau$ is future-pointing if, and only if, $\varrho'(\tau) < 0$.

6.3 Weakly trapped submanifolds in GRW spacetimes

In this section we present the main results of this chapter, which hold for the case of a weakly trapped submanifold. Nevertheless, before that we develop some computations which will be key to the proof of such results.

Let us start by considering $\psi : \Sigma \to -I \times_{\varrho} N^{n+1}$ an immersed spacelike submanifold of codimension two. We define the height function of Σ as follows.

Definition 6.5. The height function of Σ, denoted by h, is the restriction of the projection $\pi_I(t,x) = t$ to Σ, that is,

$$\begin{aligned} h : \Sigma \xrightarrow{\psi} -I \times_{\varrho} N^{n+1} \xrightarrow{\pi_I} \mathbb{R} \\ p \mapsto (\pi_I)_{|\Sigma}(p) = (\pi_I \circ \psi)(p) \end{aligned}$$

for every $p \in \Sigma$.

Recall from equation (6.9) that the gradient of π_I on $-I \times_{\varrho} N^{n+1}$ is given by $\overline{\nabla}\pi_I = -\partial_t$.Then, the gradient of h on Σ is given by

$$\nabla h = (\overline{\nabla}\pi_I)^{\top} = -\partial_t^{\top}, \tag{6.17}$$

where

$$\partial_t = \partial_t^{\top} + \partial_t^{\perp}.$$

Here $\partial_t^{\top} \in \mathfrak{X}(\Sigma)$ and $\partial_t^{\perp} \in \mathfrak{X}^{\perp}(\Sigma)$ denote, respectively, the tangential and the normal components of ∂_t.

In our computations, we will also consider the function $u = g(h)$, where $g : I \to \mathbb{R}$ is an arbitrary primitive of ϱ. Since $g' = \varrho > 0$, then $u = g(h)$ can be thought as a reparametrization of the height function. In particular, the gradient of u on Σ is

$$\nabla u = \varrho(h)\nabla h = -\varrho(h)\partial_t^{\top} = -K^{\top}, \tag{6.18}$$

where $K^{\top}$ denotes the tangential component of the closed conformal vector field K given at (6.4) along the submanifold,

$$K = K^{\top} + K^{\perp}. \tag{6.19}$$

Using (2.4) and (2.5), it follows from (6.19) that

$$\overline{\nabla}_X K = \nabla_X K^\top - \mathrm{II}(X, K^\top) + A_{K^\perp} X + \nabla^\perp_X K^\perp \tag{6.20}$$

for every $X \in \mathfrak{X}(\Sigma)$. Therefore,

$$\left(\overline{\nabla}_X K\right)^\top = \nabla_X K^\top + A_{K^\perp} X \tag{6.21}$$

and

$$\left(\overline{\nabla}_X K\right)^\perp = -\,\mathrm{II}\,(X, K^\top) + \nabla^\perp_X K^\perp. \tag{6.22}$$

On the other hand, equation (6.5) implies $\overline{\nabla}_X K = \varrho'(h) X$, so that

$$\left(\overline{\nabla}_X K\right)^\top = \varrho'(h) X \tag{6.23}$$

and

$$\left(\overline{\nabla}_X K\right)^\perp = 0. \tag{6.24}$$

Then, from (6.21) and (6.23), we see that

$$\nabla_X K^\top = \varrho'(h) X - A_{K^\perp} X, \tag{6.25}$$

and, therefore, from (6.18) we get that

$$\nabla_X \nabla u = -\nabla_X K^\top = -\varrho'(h) X + A_{K^\perp} X. \tag{6.26}$$

Finally, tracing this expression we get

$$\begin{aligned} \Delta u = &\ -n\varrho'(h) + \mathrm{tr}(A_{K^\perp}) = -n(\varrho'(h) - \langle \mathbf{H}, K\rangle) \\ = &\ -n(\varrho'(h) - \varrho(h)\langle \mathbf{H}, \partial_t\rangle). \end{aligned} \tag{6.27}$$

6.3.1 Non existence of weakly trapped submanifolds

At that point we proceed to present some non-existence results for a weakly trapped submanifolds. As a first application of (6.27) we have the following consequences for the particular case of closed submanifolds.

Lemma 6.6. Let $-I \times_\varrho N^{n+1}$ be a GRW spacetime, and let $\psi : \Sigma \to -I \times_\varrho N^{n+1}$ be a codimension two closed spacelike submanifold immersed in $-I \times_\varrho N^{n+1}$. Then, the mean curvature vector field of Σ satisfies

$$\max_\Sigma \langle \mathbf{H}, \partial_t \rangle \geq -\mathcal{H}(h_*) = \frac{\varrho'(h_*)}{\varrho(h_*)}, \tag{6.28}$$

and

$$\min_{\Sigma} \langle \mathbf{H}, \partial_t \rangle \leq -\mathcal{H}(h^*) = \frac{\varrho'(h^*)}{\varrho(h^*)}, \tag{6.29}$$

where $h_* = \min_\Sigma h$ and $h^* = \max_\Sigma h$.

Proof. Let Σ be a closed spacelike submanifold of $-I \times_\varrho N^{n+1}$, and consider the function $u = g(h)$. Since Σ is closed, the function u attains its minimum and maximum at some points $p_{\min}$ and $p_{\max}$. Since $g' = \varrho > 0$, g is strictly increasing and, at $p_{\min}$, it holds

$$u(p_{\min}) = u_* = \min_\Sigma u = g(h_*),$$

where $h_* = h(p_{\min}) = \min_\Sigma h$, and

$$\Delta u(p_{\min}) = -n(\varrho'(h_*) - \varrho(h_*)\langle \mathbf{H}, \partial_t\rangle|_{p_{\min}}) \geq 0.$$

It follows from here that

$$\max_\Sigma \langle \mathbf{H}, \partial_t \rangle \geq \langle \mathbf{H}, \partial_t \rangle|_{p_{\min}} \geq \frac{\varrho'(h_*)}{\varrho(h_*)} = -\mathcal{H}(h_*).$$

The proof of (6.29) is similar, working at $p_{\max}$. □

As an immediate outcome we can state the next corollary.

Corollary 6.7. Let $-I \times_\varrho N^{n+1}$ be a GRW spacetime.

(i) If $\mathcal{H}(t) \geq 0$, there exists no closed weakly past trapped submanifold in $-I \times_\varrho N^{n+1}$.

(ii) If $\mathcal{H}(t) \leq 0$, there exists no closed weakly future trapped submanifold in $-I \times_\varrho N^{n+1}$.

In particular, if $\mathcal{H} \equiv 0$ Corollary 6.7 implies that there exists no closed weakly trapped submanifold in $-I \times_\varrho N^{n+1}$. Actually, this case corresponds to the case where K is a timelike Killing vector field, which was already consider by Mars and Senovilla in [36, Theorem 1] in a more general context.

For a proof of Corollary 6.7, simply observe that for any weakly past trapped submanifold Σ into $-I \times_\varrho N^{n+1}$ one has $\langle \mathbf{H}, \partial_t \rangle > 0$ on Σ and, in particular, $\min_\Sigma \langle \mathbf{H}, \partial_t \rangle > 0$. Therefore, by (6.29) it must be

$$\mathcal{H}(h^*) \leq -\min_{\Sigma} \langle \mathbf{H}, \partial_t \rangle < 0.$$

Similarly, for the case of weakly future trapped submanifolds, one has

$$\max_{\Sigma} \langle \mathbf{H}, \partial_t \rangle < 0$$

and (6.28) implies

$$\mathcal{H}(h_*) \geq -\max_{\Sigma} \langle \mathbf{H}, \partial_t \rangle > 0.$$

In order to extend these results to the non-compact case, and following the terminology introduced in [2], we define the following.

Definition 6.8. We say that a spacelike submanifold $\psi : \Sigma \to -I \times_{\varrho} N^{n+1}$ is bounded away from the past infinity at height $\tau_* \in I$ if

$$\psi(\Sigma) \subset \{(t, x) \in -I \times_{\varrho} N^{n+1} : t \geq \tau_*\}.$$

In a similar way, we say that a spacelike submanifold $\psi : \Sigma \to -I \times_{\varrho} N^{n+1}$ is bounded away from the future infinity at height $\tau^* \in I$ if

$$\psi(\Sigma) \subset \{(t, x) \in -I \times_{\varrho} N^{n+1} : t \leq \tau^*\}.$$

When, in the previous definition, the height τ_* (or τ^*) is not relevant, we will say simply that Σ is bounded away from the past infinity (or from the future infinity). Finally, Σ is said to be bounded away from the infinity of $-I \times_{\varrho} N^{n+1}$ if it is bounded away from the past and future infinity; that is, if there exist $\tau_*, \tau^* \in I$, $\tau_* < \tau^*$, such that $\psi(\Sigma)$ is contained in the slab $\Omega(\tau_*, \tau^*)$ bounded between the slices N_{τ_*} and N_{τ^*}.

We are now ready to give the following result, which extends Lemma 6.6 to the non-compact case under the assumption of stochastically completeness.

Lemma 6.9. Let $-I \times_{\varrho} N^{n+1}$ be a GRW spacetime, and let $\psi : \Sigma \to -I \times_{\varrho} N^{n+1}$ be a codimension two stochastically complete spacelike submanifold immersed in $-I \times_{\varrho} N^{n+1}$.

(i) If Σ is bounded away from the past infinity, then the mean curvature vector field of Σ satisfies

$$\sup_{\Sigma} \langle \mathbf{H}, \partial_t \rangle \geq -\mathcal{H}(h_*) = \frac{\varrho'(h_*)}{\varrho(h_*)}, \tag{6.30}$$

where $h_* = \inf_\Sigma h \in I$.

(ii) If Σ is bounded away from the future infinity, then the mean curvature vector field of Σ satisfies

$$\inf_{\Sigma} \langle \mathbf{H}, \partial_t \rangle \leq -\mathcal{H}(h^*) = \frac{\varrho'(h^*)}{\varrho(h^*)}, \tag{6.31}$$

where $h^* = \sup_\Sigma h \in I$.

Proof. Let Σ be a stochastically complete spacelike submanifold of $-I \times_\varrho N^{n+1}$, and assume that Σ is bounded away from the past infinity at height $\tau_* \in I$; in particular, $h_* = \inf_\Sigma h \geq \tau_* \in I$. We start by applying the weak maximum principle (see Subsection 2.3.2) on Σ to the function $u = g(h)$, which satisfies $u_* = \inf_\Sigma u = g(h_*) > -\infty$. Therefore, there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ such that

$$u(p_k) < u_* + \frac{1}{k}, \quad \Delta u(p_k) > -\frac{1}{k}.$$

Observe that $\lim_{k\to+\infty} h(p_k) = h_*$ because g is strictly increasing. Using equation (6.27) we obtain that

$$-\frac{1}{nk} < \frac{1}{n}\Delta u(p_k) = -\varrho'(h(p_k)) + \varrho(h(p_k))\langle \mathbf{H}, \partial_t\rangle(p_k).$$

That is,

$$\langle \mathbf{H}, \partial_t\rangle(p_k) > \frac{1}{\varrho(h(p_k))}\left(\varrho'(h(p_k)) - \frac{1}{nk}\right).$$

Therefore,

$$\sup_{\Sigma} \langle \mathbf{H}, \partial_t\rangle \geq \langle \mathbf{H}, \partial_t\rangle(p_k) > \frac{1}{\varrho(h(p_k))}\left(\varrho'(h(p_k)) - \frac{1}{nk}\right).$$

Making $k \to +\infty$ here we get (6.30), since

$$\lim_{k\to+\infty} \varrho(h(p_k)) = \varrho(h_*) \text{ and } \lim_{k\to+\infty} \varrho'(h(p_k)) = \varrho'(h_*).$$

The proof of the case where Σ is bounded away from the future infinity is similar. $\square$

The following non-existence result for weakly trapped submanifolds holds directly from Lemma 6.9.

Corollary 6.10. Let $-I \times_{\varrho} N^{n+1}$ be a GRW spacetime.

(i) Let $\tau^* \in I$ and assume that $\mathcal{H}(t) > 0$ for $t \leq \tau^*$. Then there exists no stochastically complete weakly past trapped submanifold bounded away from the future infinity at height τ^*.

(ii) Let $\tau_* \in I$ and assume that $\mathcal{H}(t) < 0$ for $t \geq \tau_*$. Then there exists no stochastically complete weakly future trapped submanifold bounded away from the past infinity at height τ_*.

To prove this, simply observe that for any weakly past trapped submanifold Σ into $-I \times_{\varrho} N^{n+1}$ one has

$$\langle \mathbf{H}, \partial_t \rangle > 0$$

on Σ, which yields $\inf_\Sigma \langle \mathbf{H}, \partial_t \rangle \geq 0$. Therefore, if Σ is a stochastically complete weakly past submanifold bounded away from the future infinity, by (6.31) it must be

$$\mathcal{H}(h^*) \leq -\inf_\Sigma \langle \mathbf{H}, \partial_t \rangle \leq 0.$$

Similarly, for the case of weakly future trapped submanifolds, one has $\sup_\Sigma \langle \mathbf{H}, \partial_t \rangle \leq 0$. Therefore, if Σ is a stochastically complete weakly future submanifold bounded away from the past infinity, (6.30) implies

$$\mathcal{H}(h_*) \geq -\sup_\Sigma \langle \mathbf{H}, \partial_t \rangle \geq 0.$$

6.3.2 Rigidity of marginally trapped submanifolds

In this subsection we derive ss; equivalently $(\log \varrho)'' \leq 0$. This hypothesis has been widely used by several authors to obtain rigidity results for spacelike hypersurfaces with constant mean curvature in GRW spacetimes and, as we have seen in Section 6.1 it is closely related to the TCC.

The first of our rigidity results holds for closed marginally future trapped submanifolds and states as follows.

Theorem 6.11. Let $-I \times_{\varrho} N^{n+1}$ be a GRW spacetime such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$), and let $\psi : \Sigma \to -I \times_{\varrho} N^{n+1}$ be a codimension two closed marginally future trapped submanifold immersed in $-I \times_{\varrho} N^{n+1}$. Then

$$\min_\Sigma \|\mathbf{H}_0\| \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \max_\Sigma \|\mathbf{H}_0\|, \qquad (6.32)$$

where $h_* = \min_\Sigma h$, $h^* = \max_\Sigma h$, and $\mathbf{H}_0$ stands for the spacelike component of the lightlike vector $\mathbf{H}$ which is orthogonal to ∂_t. As a consequence, if $\|\mathbf{H}_0\|$

is constant then $\psi(\Sigma)$ is contained in a slice $\{\tau\}\times N$ with $\varrho'(\tau)<0$, and $\Sigma=\{\tau\}\times\Sigma_0$ with $\Sigma_0\subset N$ a closed hypersurface of constant mean curvature $|H|=-\varrho'(\tau)>0$.

Proof. The mean curvature vector $\mathbf{H}$ of Σ decomposes as

$$\mathbf{H}=\mathbf{H}_0-\langle\mathbf{H},\partial_t\rangle\partial_t, \tag{6.33}$$

and then

$$\langle\mathbf{H},\mathbf{H}\rangle=\|\mathbf{H}_0\|^2-\langle\mathbf{H},\partial_t\rangle^2. \tag{6.34}$$

Assume that Σ is marginally future trapped. In that case $\langle\mathbf{H},\mathbf{H}\rangle=0$ and

$$\langle\mathbf{H},\partial_t\rangle=-\|\mathbf{H}_0\|<0.$$

Therefore, it follows from Lemma 6.6 that

$$\max_\Sigma\|\mathbf{H}_0\|=-\min_\Sigma\langle\mathbf{H},\partial_t\rangle\geq\mathcal{H}(h^*) \tag{6.35}$$

and

$$\min_\Sigma\|\mathbf{H}_0\|=-\max_\Sigma\langle\mathbf{H},\partial_t\rangle\leq\mathcal{H}(h_*). \tag{6.36}$$

Since $\mathcal{H}'(t)\geq 0$, the function $\mathcal{H}(t)$ is non-decreasing and $\mathcal{H}(h_*)\leq\mathcal{H}(h^*)$, which jointly with (6.35) and (6.36) gives (6.32).

As a consequence, if $\|\mathbf{H}_0\|$ is constant we know from (6.32) that

$$\mathcal{H}(h_*)=\mathcal{H}(h^*)=\|\mathbf{H}_0\|=\text{ constant.}$$

Since $\mathcal{H}(t)$ is non-decreasing, it follows from here that $\mathcal{H}(t)=\|\mathbf{H}_0\|=\text{constant}$ on $[h_*,h^*]$. That is,

$$\mathcal{H}(h)=\|\mathbf{H}_0\|=-\langle\mathbf{H},\partial_t\rangle\quad\text{on }\Sigma.$$

In other words, $\varrho'(h)-\varrho(h)\langle\mathbf{H},\partial_t\rangle=0$ on Σ, which by (6.27) implies $\Delta u=0$ on Σ. That is, u is a harmonic function on Σ, which is a closed manifold. Hence, $u=g(h)$ is constant on Σ, and since $g(t)$ is increasing this means that h itself is constant on Σ; that is, $\psi(\Sigma)$ is contained in a slice $\{\tau\}\times N$. As we have already seen in Subsection 6.1.1, since $\psi:\Sigma\to -I\times_\varrho N^{n+1}$ is a marginally future trapped submanifold it then follows that $\varrho'(\tau)<0$ and the projection $\phi=\pi_N\circ\psi:\Sigma\to N^{n+1}$ is a hypersurface with constant mean curvature $|H|=-\varrho'(\tau)>0$. $\square$

In the same way, for the case of closed marginally past trapped submanifolds we have the following.

Theorem 6.12. Let $-I \times_{\varrho} N^{n+1}$ be a GRW spacetime such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$), and let $\psi : \Sigma \to -I \times_{\varrho} N^{n+1}$ be a codimension two closed marginally past trapped submanifold immersed in $-I \times_{\varrho} N^{n+1}$. Then

$$\min_{\Sigma} \|\mathbf{H}_0\| \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \max_{\Sigma} \|\mathbf{H}_0\|,$$

where $h_* = \min_{\Sigma} h$, $h^* = \max_{\Sigma} h$, and $\mathbf{H}_0$ stands for the spacelike component of the lightlike vector $\mathbf{H}$ which is orthogonal to ∂_t. As a consequence, if $\|\mathbf{H}_0\|$ is constant then $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times N$ with $\varrho'(\tau) > 0$, and $\Sigma = \{\tau\} \times \Sigma_0$ with $\Sigma_0 \subset N$ a closed hypersurface of constant mean curvature $|H| = \varrho'(\tau) > 0$.

The proof of Theorem 6.12 is similar to that of Theorem 6.11 simply by observing that since Σ is marginally past trapped, then $\langle \mathbf{H}, \partial_t \rangle = \|\mathbf{H}_0\| > 0$ on Σ.

In particular, we obtain the following consequence.

Corollary 6.13. Let $-I \times_{\varrho} N^{n+1}$ be a GRW spacetime such that $\mathcal{H}(t) \neq 0$ and $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$), and let $\psi : \Sigma \to -I \times_{\varrho} N^{n+1}$ be a codimension two closed marginally trapped submanifold immersed in $-I \times_{\varrho} N^{n+1}$.

(i) If $\mathcal{H}(t) \geq 0$, then Σ is necessarily marginally future trapped and

$$\min_{\Sigma} \|\mathbf{H}_0\| \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \max_{\Sigma} \|\mathbf{H}_0\|.$$

(ii) If $\mathcal{H}(t) \leq 0$, then Σ is necessarily marginally past trapped and

$$\min_{\Sigma} \|\mathbf{H}_0\| \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \max_{\Sigma} \|\mathbf{H}_0\|.$$

Therefore, if $\|\mathbf{H}_0\|$ is constant then $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times N$ with $\mathcal{H}(\tau) \neq 0$, and $\Sigma = \{\tau\} \times \Sigma_0$ with $\Sigma_0 \subset N$ a closed hypersurface of constant mean curvature $|H| = |\varrho'(\tau)| > 0$.

Proof. In case (i), using Corollary 6.7, we obtain that Σ is necessarily future trapped and we may apply directly Theorem 6.11. Similarly, in case (ii) Σ is necessarily past trapped by Corollary 6.7 and the result then follows from Theorem 6.12. □

The next result extends Theorem 6.11 to the case of stochastically complete submanifolds.

Theorem 6.14. Let $-I \times_\varrho N^{n+1}$ be a GRW spacetime such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$). Let $\psi : \Sigma \to -I \times_\varrho N^{n+1}$ be a codimension two stochastically complete marginally future trapped submanifold which is bounded away from the infinity. Then

$$\inf_\Sigma \|\mathbf{H}_0\| \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \sup_\Sigma \|\mathbf{H}_0\|, \tag{6.37}$$

where $h_* = \inf_\Sigma h$, $h^* = \sup_\Sigma h$, and $\mathbf{H}_0$ stands for the spacelike component of the lightlike vector $\mathbf{H}$ which is orthogonal to ∂_t. As a consequence, if $\|\mathbf{H}_0\|$ is constant and $\mathcal{H}(t)$ is not locally constant (in other words, the equality $\mathcal{H}'(t) = 0$ holds only at isolated points of I), then $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times N$ with $\mathcal{H}(\tau) > 0$ and $\Sigma = \{\tau\} \times \Sigma_0$ with $\Sigma_0 \subset N$ a hypersurface of constant mean curvature $|H| = -\varrho'(\tau) > 0$.

Proof. The proof of (6.37) is similar to that of (6.32) in Theorem 6.11, but using now Lemma 6.9 instead of Lemma 6.6. As a consequence, if $\|\mathbf{H}_0\|$ is constant we know from formula (6.37) that

$$\mathcal{H}(h^*) = \mathcal{H}(h_*) = \|\mathbf{H}_0\| = \text{ constant.}$$

The hypothesis on the function $\mathcal{H}(t)$ implies now that it is strictly increasing on I. Therefore $h_* = h^*$ and h is constant on Σ; that is, $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times N$. As we have already seen in Subsection 6.1.1, since $\psi : \Sigma \to -I \times_\varrho N^{n+1}$ is marginally future trapped it then follows that $\mathcal{H}(\tau) > 0$ and the projection $\phi = \pi_N \circ \psi : \Sigma \to N^{n+1}$ is a hypersurface with constant mean curvature $|H| = -\varrho'(\tau) > 0$. $\square$

Similarly we have the next results, which extend Theorem 6.12 and Corollary 6.13 to the stochastically complete case.

Theorem 6.15. Let $-I \times_\varrho N^{n+1}$ be a GRW spacetime such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$). Let $\psi : \Sigma \to -I \times_\varrho N^{n+1}$ be a codimension two stochastically complete marginally past trapped submanifold which is bounded away from the infinity. Then

$$\inf_\Sigma \|\mathbf{H}_0\| \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \sup_\Sigma \|\mathbf{H}_0\|,$$

where $h_* = \inf_\Sigma h$, $h^* = \sup_\Sigma h$, and $\mathbf{H}_0$ stands for the spacelike component of the lightlike vector $\mathbf{H}$ which is orthogonal to ∂_t. As a consequence, if $\|\mathbf{H}_0\|$ is constant and $\mathcal{H}(t)$ is not locally constant (in other words, the equality $\mathcal{H}'(t) = 0$ holds only at isolated points of I), then $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times N$ with $\mathcal{H}(\tau) < 0$ and $\Sigma = \{\tau\} \times \Sigma_0$ with $\Sigma_0 \subset N$ a hypersurface of constant mean curvature $|H| = \varrho'(\tau) > 0$.

Corollary 6.16. Let $-I \times_\varrho N^{n+1}$ be a GRW spacetime such that $\mathcal{H}(t) \neq 0$ and $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$), and let $\psi : \Sigma \to -I \times_\varrho N^{n+1}$ be a codimension two stochastically complete marginally trapped submanifold which is bounded away from the infinity.

(i) If $\mathcal{H}(t) > 0$, then Σ is necessarily marginally future trapped and

$$\inf_\Sigma \|\mathbf{H}_0\| \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \sup_\Sigma \|\mathbf{H}_0\|.$$

(ii) If $\mathcal{H}(t) < 0$, then Σ is necessarily marginally past trapped and

$$\inf_\Sigma \|\mathbf{H}_0\| \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \sup_\Sigma \|\mathbf{H}_0\|.$$

As a consequence, if $\|\mathbf{H}_0\|$ is constant and $\mathcal{H}(t)$ is not locally constant (in other words, the equality $\mathcal{H}'(t) = 0$ holds only at isolated points of I) then $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times N$ and $\Sigma = \{\tau\} \times \Sigma_0$ with $\Sigma_0 \subset N$ a hypersurface of constant mean curvature $|H| = |\varrho'(\tau)| > 0$.

Remark 6.17. Recall from Subsection 2.3.3 that every parabolic Riemannian manifold is stochastically complete. Under the assumption of parabolicity, our previous results for stochastically complete submanifolds can be improved by removing the hypothesis on the non-locally constancy of the function $\mathcal{H}(t)$. Observe that this extra hypothesis was not assumed in the case of closed submanifolds. For instance, in Theorem 6.14, to see why in the parabolic case we do not need this extra hypothesis neither, observe that once we know that $\mathcal{H}(h_*) = \mathcal{H}(h^*) = \|\mathbf{H}_0\| =$ constant we can conclude, as in the closed case, that $\varrho'(h) - \varrho(h)\langle \mathbf{H}, \partial_t \rangle = 0$ on Σ and, from (6.27), u is a bounded harmonic function on Σ. But Σ being parabolic, this implies that u must be constant and, equivalently, h must be constant.

6.4 Applications

Finally in this chapter, we apply the results of previous section to some specific GRW spacetimes. Let us consider first the case where the Riemannian fiber is the $(n+1)$-dimensional Euclidean space. When $n = 2$ we have the following rigidity result for immersed topological 2-spheres in $-I \times_{\varrho} \mathbb{R}^3$.

Theorem 6.18. Let $-I \times_{\varrho} \mathbb{R}^3$ be a RW spacetime whose fiber is the Euclidean 3-space and such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$). The only marginally trapped topological 2-spheres which are immersed in $-I \times_{\varrho} \mathbb{R}^3$ with $\|\mathbf{H}_0\|$ constant are the embedded spheres given as $\{\tau\} \times \mathbb{S}^2(r_\tau)$, with $r_\tau = 1/|\varrho'(\tau)|$ for every $\tau \in I$ with $\varrho'(\tau) \neq 0$.

Proof. Let Σ be a marginally trapped topological 2-sphere immersed in $-I \times_{\varrho} \mathbb{R}^3$ with $\|\mathbf{H}_0\|$ constant. By Theorems 6.11 and 6.12, we know that $\Sigma = \{\tau\} \times \Sigma_0$ where $\varrho'(\tau) \neq 0$ and Σ_0 is an immersed topological 2-sphere in $\mathbb{R}^3$ with constant mean curvature $H^2 = \varrho'(\tau)^2 > 0$. By Hopf's theorem [25], it is well known that when $n = 2$ the only immersed topological 2-spheres in $\mathbb{R}^3$ with constant mean curvature are the embedded round spheres $\mathbb{S}^2(r)$ of radius $r > 0$, with $H^2 = 1/r^2 > 0$. Therefore $\Sigma_0 = \mathbb{S}^2(r_\tau)$ with $r_\tau = 1/|\varrho'(\tau)|$, and this finishes the proof. $\square$

Theorem 6.18 includes some cases of physical relevance. For instance, it is not difficult to see that the Lorentzian warped product $-\mathbb{R} \times_{e^t} \mathbb{R}^{n+1}$ models the $(n+2)$-dimensional steady state spacetime, defined in Section 4.4. To see it, choose, for instance, $\mathbf{a} = (1, 0, \ldots, 0, 1) \in \mathbb{L}^{n+3}$. Then the map $\Phi : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{S}_1^{n+2}$ given by

$$\Phi(t, p) = (-\sinh(t) - \frac{e^t|p|^2}{2}, e^t p, \cosh(t) - \frac{e^t|p|^2}{2})$$

defines an isometry between $-\mathbb{R} \times_{e^t} \mathbb{R}^{n+1}$ and the $(n+2)$-dimensional steady state spacetime. Using this and Theorem 6.18 we obtain the following.

Corollary 6.19. Let $-\mathbb{R} \times_{e^t} \mathbb{R}^3$ be the steady state 4-spacetime. The only marginally trapped topological 2-spheres which are immersed in the steady state 4-spacetime with $\|\mathbf{H}_0\|$ constant are the (necessarily marginally past trapped) embedded spheres given as $\{\tau\} \times \mathbb{S}^2(r_\tau)$, for every $\tau \in \mathbb{R}$, where $r_\tau = e^{-\tau}$.

On the other hand, when $I = (0, +\infty)$ and $\varrho(t) = t^{2/3}$, $-(0, +\infty) \times_{t^{2/3}} \mathbb{R}^3$ is called the Einstein-de Sitter spacetime and it corresponds to the Friedmann cosmological model with flat Riemannian fiber.

Corollary 6.20. Let $-(0, +\infty) \times_{t^{2/3}} \mathbb{R}^3$ be the Einstein-de Sitter spacetime. The only marginally trapped topological 2-spheres which are immersed in the Einstein-de Sitter spacetime with $\|\mathbf{H}_0\|$ constant are the (necessarily marginally past trapped) embedded spheres given as $\{\tau\} \times \mathbb{S}^2(r_\tau)$, for every $\tau > 0$, where $r_\tau = 3\tau^{1/3}/2$.

The general n-dimensional version of Theorem 6.18 is based on the well-known Alexandrov's theorem which states that the only closed embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature are the embedded round spheres $\mathbb{S}^n(r)$ of radius $r > 0$, with $H^2 = 1/r^2 > 0$. Then, as a consequence of Theorems 6.11 and 6.12 we have the following n-dimensional version of Theorem 6.18.

Theorem 6.21. Let $-I \times_{\varrho} \mathbb{R}^{n+1}$ be a RW spacetime whose fiber is the Euclidean $(n+1)$-space and such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$). The only closed marginally trapped n-submanifolds which are embedded in $-I \times_{\varrho} \mathbb{R}^{n+1}$ with $\|\mathbf{H}_0\|$ constant are the embedded n-spheres given as $\{\tau\} \times \mathbb{S}^n(r_\tau)$, with $r_\tau = 1/|\varrho'(\tau)|$ for every $\tau \in I$ with $\varrho'(\tau) \neq 0$.

The proof of Theorem 6.21 is similar to that of Theorem 6.18, observing now that Σ_0 is a closed embedded hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature. In particular, Corollary 6.19 and Corollary 6.20 have the following n-dimensional version.

Corollary 6.22. Let $-\mathbb{R} \times_{e^t} \mathbb{R}^{n+1}$ be the steady state spacetime. The only closed marginally trapped n-submanifolds which are embedded in the steady state spacetime with $\|\mathbf{H}_0\|$ constant are the (necessarily marginally past trapped) embedded n-spheres given as $\{\tau\} \times \mathbb{S}^n(r_\tau)$, for every $\tau \in \mathbb{R}$, where $r_\tau = e^{-\tau}$.

Corollary 6.23. Let $-(0, +\infty) \times_{t^{2/3}} \mathbb{R}^{n+1}$ be the Einstein-de Sitter spacetime. The only closed marginally trapped n-submanifolds which are embedded in the Einstein-de Sitter spacetime with $\|\mathbf{H}_0\|$ constant are the (necessarily marginally past trapped) embedded n-spheres given as $\{\tau\} \times \mathbb{S}^n(r_\tau)$, for every $\tau > 0$, where $r_\tau = 3\tau^{1/3}/2$.

Remark 6.24. In the results given for closed marginally trapped n-submanifolds in the steady state $(n+2)$-dimensional spacetime (Corollary 6.19 for immersed topological 2-spheres and Corollary 6.22 for embedded closed n-submanifolds) we obtain as a consequence that they are necessarily past trapped. The same happens for the case when the ambient space is the Einstein-de Sitter spacetime. Actually, since in theses examples $\mathcal{H}(t) < 0$ for every $t \in I$, we know from Corollary 6.7 that there exists no closed weakly future trapped submanifolds in such ambient spaces. Even more, from Corollary 6.10 we know that there exists no stochastically complete weakly future trapped submanifolds bounded away from the past infinity neither in the steady state spacetime nor in the Einstein-de Sitter spacetime.

On the other hand, Hopf's theorem holds also for topological 2-spheres in the 3-dimensional sphere $\mathbb{S}^3$ and in the 3-dimensional hyperbolic space $\mathbb{H}^3$, yielding the following results.

Theorem 6.25. Let $-I \times_\varrho \mathbb{S}^3$ be a RW spacetime whose fiber is the 3-dimensional sphere and such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$). The only marginally trapped topological 2-spheres which are immersed in $-I \times_\varrho \mathbb{S}^3$ with $\|\mathbf{H}_0\|$ constant are the embedded spheres given as $\{\tau\} \times \mathbb{S}^2(r_\tau)$, with $r_\tau = 1/\sqrt{1 + \varrho'(\tau)^2}$ for every $\tau \in I$ with $\varrho'(\tau) \neq 0$.

The proof of Theorem 6.25 is similar to that of Theorem 6.18, observing now that the only immersed topological 2-spheres in $\mathbb{S}^3$ with constant mean curvature $H \neq 0$ are the round spheres $\mathbb{S}^2(r)$ with $0 < r < 1$, having $H^2 = (1 - r^2)/r^2$. Therefore in this case $r_\tau = 1/\sqrt{1 + \varrho'(\tau)^2}$.

On the other hand, Hopf's theorem in $\mathbb{H}^3$ implies that the only immersed topological 2-spheres in $\mathbb{H}^3$ with constant mean curvature are the round spheres $\mathbb{S}^2(r)$ with $r > 0$, having $H^2 = (1 + r^2)/r^2 > 1$. Therefore in this case it must be $\varrho'(\tau)^2 > 1$ and $r_\tau = 1/\sqrt{\varrho'(\tau)^2 - 1}$, and we obtain the following.

Theorem 6.26. Let $-I \times_\varrho \mathbb{H}^3$ be a RW spacetime whose fiber is the 3-dimensional hyperbolic space and such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$). The only marginally trapped topological 2-spheres which are immersed in $-I \times_\varrho \mathbb{H}^3$ with $\|\mathbf{H}_0\|$ constant are the embedded spheres given as $\{\tau\} \times \mathbb{S}^2(r_\tau)$, with $r_\tau = 1/\sqrt{\varrho'(\tau)^2 - 1}$ for every $\tau \in I$ with $\varrho'(\tau)^2 - 1 > 0$.

Theorem 6.26 includes the case where $I = (0, +\infty)$ and $\varrho(t) = \sinh(t)$. In this case $-(0, +\infty) \times_{\sinh(t)} \mathbb{H}^3$ models the open region of the 4-dimensional de Sitter space $\mathbb{S}^4_1$ given by

$$\{x \in \mathbb{S}^4_1 : \langle x, \mathbf{a} \rangle > 1\}$$

where $\mathbf{a} \in \mathbb{R}^5_1$ is a unit spacelike vector. To see it in general dimension $n \geq 2$, choose, for instance, $\mathbf{a} = (0, \ldots, 0, 1) \in \mathbb{R}_1^{n+3}$ and consider

$$\mathbb{H}^{n+1} = \{p \in \mathbb{L}^{n+2} : \langle p, p \rangle = -1\}.$$

Then the map $\Phi : (0, +\infty) \times \mathbb{H}^{n+1} \to \mathbb{S}_1^{n+2}$ given by

$$\Phi(t, p) = \cosh(\log(1+\sqrt{2}) - t)(p, \sqrt{2}) - \sinh(\log(1+\sqrt{2}) - t)(\sqrt{2}p, 1)$$

defines an isometry between the Lorentzian warped product $-(0, +\infty) \times_{\sinh(t)} \mathbb{H}^{n+1}$ and the open region of $\mathbb{S}_1^{n+2}$ defined by

$$\{x \in \mathbb{S}_1^{n+2} : \langle x, \mathbf{a} \rangle > 1\}, \quad \mathbf{a} = (0, \ldots, 0, 1).$$

Therefore, Theorem 6.26 gives the following.

Corollary 6.27. Let $-(0, +\infty) \times_{\sinh(t)} \mathbb{H}^3$ be the open region of the 4-dimensional de Sitter space defined by $\{x \in \mathbb{S}_1^4 : \langle x, \mathbf{a} \rangle > 1\}$, where $\mathbf{a} \in \mathbb{L}^5$ is a unit spacelike vector. The only marginally trapped topological 2-spheres which are immersed in this region with $\|\mathbf{H}_0\|$ constant are the (necessarily marginally past trapped) embedded spheres given as $\{\tau\} \times \mathbb{S}^2(r_\tau)$, with $r_\tau = 1/\sinh(\tau)$ for every $\tau > 0$.

As in the case of Theorem 6.18, Theorem 6.26 also has an n-dimensional version which is based in the corresponding Alexandrov's theorem. Actually, when the Riemannian fiber is the hyperbolic space, Alexandrov's theorem states that the only closed embedded hypersurfaces in $\mathbb{H}^{n+1}$ with constant mean curvature H are the embedded round spheres $\mathbb{S}^n(r)$ of radius $r > 0$, having $H^2 = (1+r^2)/r^2 > 1$. Therefore, the n-dimensional versions of Theorem 6.26 and Corollary 6.27 are as follows.

Theorem 6.28. Let $-I \times_\varrho \mathbb{H}^{n+1}$ be a RW spacetime whose fiber is the $(n+1)$-dimensional hyperbolic space and such that $\mathcal{H}'(t) \geq 0$ (equivalently $(\log \varrho)'' \leq 0$). The only closed marginally trapped n-submanifolds which are embedded in $-I \times_\varrho \mathbb{H}^{n+1}$ with $\|\mathbf{H}_0\|$ constant are the embedded n-spheres given as $\{\tau\} \times \mathbb{S}^n(r_\tau)$, with $r_\tau = 1/\sqrt{\varrho'(\tau)^2 - 1}$ for every $\tau \in I$ with $\varrho'(\tau)^2 - 1 > 0$.

Corollary 6.29. Let $-(0, +\infty) \times_{\sinh(t)} \mathbb{H}^{n+1}$ be the open region of the $(n+2)$-dimensional de Sitter space defined by $\{x \in \mathbb{S}_1^{n+2} : \langle x, \mathbf{a} \rangle > 1\}$, where $\mathbf{a} \in \mathbb{L}^{n+3}$ is a unit spacelike vector. The only closed marginally trapped n-submanifolds which are embedded in this region with $\|\mathbf{H}_0\|$ constant are the (necessarily marginally past trapped) embedded n-spheres given as $\{\tau\} \times \mathbb{S}^n(r_\tau)$, with $r_\tau =$

$1/\sinh(\tau)$ for every $\tau > 0$.

Remark 6.30. Similarly as in Remark 6.24, when the ambient spacetime is the open region of de Sitter space modeled as $-(0,+\infty)\times_{\sinh(t)}\mathbb{H}^{n+1}$, since $\mathcal{H}(t) < 0$ for every $t \in (0,+\infty)$, we also obtain there exists no stochastically complete weakly future trapped submanifolds bounded away from the past infinity.

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Durante la realizacin del trabajo recogido en esta tesis y la redaccin de la misma la autora ha contado con la financiación de la Consejería de Industria, Turismo, Empresa e Innovación de la CARM a través de la Fundación Séneca por medio de una ayuda del Programa Séneca 19783/FPI/15.